Trigonometry Review

Positive and Negative Rotations: A rotation is said to be positive if the initial ray is rotated counter-clockwise to the terminal ray and said to be negative if the initial ray is rotated clockwise to the terminal ray.

Coterminal Angles: Any two angles drawn in standard position that share a terminal ray.

Reference Angles: The positive acute angle formed by the terminal ray and the x-axis.

**Exercise #1:** Give a negative angle that is coterminal with each of the following positive angles, \( \alpha \).

(a) \( \alpha = 90° \)

(b) \( \alpha = 330° \)

**Exercise #2:** For each of the following angles, beta, draw a rotation diagram and then state beta's reference angle, \( \beta_r \).

(a) \( \beta = 160° \)

(b) \( \beta = 300° \)

\[ \beta_r = 180° - 160° = 20° \]

\[ \beta_r = 360° - 300° = 60° \]

**Signs of the Quadrants**

SIN (CSC) ONLY +

TAN (COT) ONLY +

Cos (SEC) ONLY +

All FUNCTIONS +

In what Quadrant does \( \theta \) terminate if 

\( \sin \theta < 0 \) and \( \tan \theta < 0 \)?

\( (\text{III}) \) and \( (\text{IV}) \)

Unit 9 Summary

Radian Measurement

**Degrees \( \rightarrow \) Radians**

Multiply by \( \frac{\pi}{180°} \)

**Radians \( \rightarrow \) Degrees**

Multiply by \( \frac{180°}{\pi} \)

**Examples:**

\[ 150° \left( \frac{\pi}{180°} \right) = \frac{150\pi}{180} = \frac{5\pi}{6} \]

\[ 360° \left( \frac{180°}{\pi} \right) = 270° \]
The Unit Circle
Every coordinate can be expressed in rectangular form: \((x, y)\) or in trigonometric (polar) form: \((x, y) \rightarrow (\cos \theta, \sin \theta)\)

Pythagorean Identity: For any angle, \(\theta\), \((\cos \theta)^2 + (\sin \theta)^2 = 1\)

Exercise #1: An angle, \(\alpha\), has a terminal ray that falls in the second quadrant. If it is known that \(\sin(\alpha) = \frac{3}{5}\), determine the value of \(\cos(\alpha)\).

\[
\left(\cos \alpha\right)^2 + \left(\frac{3}{5}\right)^2 = 1 \Rightarrow \left(\cos \alpha\right)^2 + \frac{9}{25} = 1
\]

\[
\left(\cos \alpha\right)^2 = \frac{16}{25} \Rightarrow \cos \alpha = \pm\frac{4}{5} = \pm\frac{4}{5}
\]

Since the terminal ray of \(\alpha\) lies in the second quadrant, its cosine must be negative, thus:

\[
\cos(\alpha) = -\frac{4}{5}
\]
Reciprocal Trig Functions

SECANT: \( \sec(\theta) = \frac{1}{\cos(\theta)} \)

(Recall that: \( \tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)} \))

COSECANT: \( \csc(\theta) = \frac{1}{\sin(\theta)} \)

COTANGENT: \( \cot(\theta) = \frac{1}{\tan(\theta)} \) or equivalently \( \cot(\theta) = \frac{\cos(\theta)}{\sin(\theta)} \)

Exercise #1:
(a) \( \sec(60^\circ) \)

\[
= \frac{1}{\cos(60^\circ)} = \frac{1}{\frac{1}{2}} = 2
\]

(b) \( \cot(150^\circ) \)

\[
= \frac{\cos(150^\circ)}{\sin(150^\circ)} = \frac{-\sqrt{3}/2}{1/2} = -\sqrt{3}/2
\]

(c) \( \csc\left(\frac{3\pi}{4}\right) \)

\[
= \frac{1}{\sin\left(\frac{3\pi}{4}\right)} = \frac{1}{\frac{\sqrt{2}}{2}} = \sqrt{2}
\]

Arc Length
\[ S = \theta \cdot r \quad S = \text{Arc Length} \quad \theta = \text{central angle (MUST be in radians!)} \quad r = \text{radius} \]

Exercise #2: Use the formula above to answer each of the following.
(a) Determine the number of radians that the minute hand of a clock passes through if it has a length of 5 inches and its tip travels a total distance of 13 inches.

In this case \( r = 5 \) and \( s = 13 \) so:

\[ \theta = \frac{13}{5} \text{ or } 2.6 \]

(b) If a pendulum swings through an angle of 0.55 radians, what distance does its tip travel if it has a length of 8 feet?

In this case \( r = 8 \) and \( \theta = 0.55 \) so:

\[ s = (0.55)(8) = 4.4 \text{ ft} \]

Modeling with Trig Graphs

For \( y = A \sin(Bx) + C \) and \( y = A \cos(Bx) + C \)

| \( A \) | the amplitude or distance the sinusoidal model rises and falls above its midline |
| \( C \) | the midline or average \( y \)-value of the sinusoidal model (Avg of Max & Min) |
| \( B \) | the frequency of the sinusoidal model – related to the period, \( P \), by the equation \( B = \frac{2\pi}{P} \) |
| \( P \) | the period of the sinusoidal model – the minimum distance along the \( x \)-axis for the cycle to repeat |
Example 1:

On a standard summer day in upstate New York, the temperature outside can be modeled using the sinusoidal equation \( O(t) = 11 \cos \left( \frac{\pi}{12} t \right) + 71 \), where \( t \) represents the number of hours since the peak temperature for the day.

(a) Sketch a graph of this function on the axes below for one day.

(b) For \( 0 \leq t \leq 24 \), graphically determine all points in time when the outside temperature is equal to 75 degrees. Round your answers to the nearest tenth of an hour.

According to our graph at the left, the temperature outside is 75 °F at:
\[ t = 4.6 \text{ and } 19.4 \text{ hours} \]

Example 2:

Evie is on a swing thinking about trigonometry (no seriously!). She realizes that her height above the ground is a periodic function of time that can be modeled using \( h = 3 \cos \left( \frac{\pi}{2} t \right) + 5 \), where \( t \) represents time in seconds.

Which of the following is the range of Evie’s heights?

1. \( 2 \leq h \leq 8 \)  
2. \( 4 \leq h \leq 8 \)  
3. \( 3 \leq h \leq 5 \)  
4. \( 2 \leq h \leq 5 \)

\[ h_{\text{min}} = 5 - 3 = 2 \text{ ft} \]  
\[ h_{\text{max}} = 5 + 3 = 8 \text{ ft} \]

Basic Right Triangle Trigonometry.

**SOH CAH TOA**

\[
\sin \theta = \frac{\text{opposite}}{\text{hypotenuse}} \quad \cos \theta = \frac{\text{adjacent}}{\text{hypotenuse}} \quad \tan \theta = \frac{\text{side opposite}}{\text{side adjacent}}
\]
1. Write the function with the following parameters:

<table>
<thead>
<tr>
<th>Function</th>
<th>Amplitude</th>
<th>Period</th>
<th>Horizontal</th>
<th>Vertical Shift</th>
<th>Equation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cosine</td>
<td>0.6</td>
<td>$4\pi$</td>
<td>None</td>
<td>None</td>
<td>$y = 0.6\cos\left(\frac{1}{2}x\right)$</td>
</tr>
<tr>
<td>Sine</td>
<td>5</td>
<td>$2\pi$</td>
<td>None</td>
<td>Up 2</td>
<td>$y = 5\sin\left(3x\right) + 2$</td>
</tr>
<tr>
<td>Cosine</td>
<td>3</td>
<td>$4\pi$</td>
<td>Left $\frac{\pi}{2}$</td>
<td>Down 10</td>
<td>$y = 3\cos\left(\frac{1}{3}(x+\frac{\pi}{2})\right) - 10$</td>
</tr>
<tr>
<td>Sine</td>
<td>$\frac{2}{5}$</td>
<td>$\pi$</td>
<td>Right $\frac{\pi}{3}$</td>
<td>none</td>
<td>$y = \frac{2}{5}\sin\left(6(x-\frac{\pi}{3})\right)$</td>
</tr>
</tbody>
</table>

2. Write the equation of each curve as a sine and cosine function

a) 

- **Sine:** $y = -4\sin\left(2(x-45^\circ)\right) + 1$
- **Cosine:** $y = 4\cos(2x) + 1$
  
  - **Amplitude:** 4
  - **Frequency:** $\frac{\pi}{2}$
  - **Midline:** 1
  - **Permiter:** $180^\circ$ or $\pi$

b) 

- **Sine:** $y = 2.4\sin\left(\frac{\pi}{2}x\right)$
- **Cosine:** $y = 2.4\cos\left(\frac{\pi}{2}(x-1)\right)$

  - **Amplitude:** 2.4
  - **Midline:** 0
  - **Frequency:** $\frac{2\pi}{4} = \frac{\pi}{2}$
3. The period of the function with rule \( f(x) = 3 \cos(2\pi x) \) is:

\[
\text{Per} = \frac{2\pi}{2\pi} = 1
\]

4. The minimum value of \( 3 + 2\cos x \) is:

\[
\text{Min.} = \text{Mid.} - \text{AMP} \Rightarrow 3 - 2 = 1
\]

\[
\text{Mid.} = 3
\]

\[
\text{AMP} = 2
\]

5. An angle of \( \frac{2\pi}{7} \) radians expressed in degrees (correct to two decimal places) is:

\[
\frac{2(180)}{7} = 51.43^\circ
\]

6. The least positive value of \( x \) for which the graph of \( y = 2\cos 2x + 2 \) touches the \( x \)-axis is:

\[
\begin{align*}
\text{A} & \quad x = \frac{\pi}{4} \\
\text{B} & \quad x = \frac{\pi}{2} \\
\text{C} & \quad x = \frac{3\pi}{4} \\
\text{D} & \quad x = \frac{3\pi}{2} \\
\text{E} & \quad x = \pi
\end{align*}
\]

7. If \( \sin x = -1 \) and \( x \in [0, 2\pi] \), the value of \( x \) is:

\[
\begin{align*}
\text{A} & \quad 0 \\
\text{B} & \quad \frac{3\pi}{2} \\
\text{C} & \quad \frac{\pi}{4} \\
\text{D} & \quad \pi \\
\text{E} & \quad \frac{\pi}{2}
\end{align*}
\]

\[
\sin \left( \frac{3\pi}{2} \right) = -1
\]
8. The equation $2 \sin x + 1 = b$, where $b$ is a positive real number, has one solution in the interval $(0, 2\pi)$. The value of $b$ is:

A 1  
B 1.5  
C 2  
D 3  
E 4

9. The function $f(x) = -2 \sin (x) + 2$ has range and amplitude respectively of:

A $[-2, 2]$ and 2  
B $[0, 2]$ and 4  
C $[0, 2]$ and 2  
D $[0, 4]$ and 2  
E $[-2, 0]$ and 4

10. The function $f(x) = a \cos (bx)$, where $a$, $b$ and $c$ are positive constants, has period:

A $a$  
B $b$  
C $\frac{2\pi}{a}$  
D $\frac{2\pi}{b}$  
E $\frac{b}{2\pi}$

11. The vertical distance from the ground of a point on a wheel as it rotates is given by $D(t) = 4 - 4 \sin (\pi t)$, where $t$ is the time in minutes. What is the time, in seconds, for a full rotation of the wheel?

A 10  
B 15  
C 30  
D 60  
E 90

\[ \frac{1}{4} \text{ of a min} = \frac{1}{4} (60) = 15 \text{ seconds} \]
12. Show all work and round to the nearest tenth.

a) \[ \cos 41^\circ = \frac{x}{25} \]
   \[ x = 25 \cos 41^\circ \]
   \[ x \approx 18.9 \]

b) \[ \tan 54^\circ = \frac{x}{\sqrt{5}} \]
   \[ x = 5\sqrt{5} \tan 54^\circ \]
   \[ x \approx 61.9 \]

13. Show all work and round to the nearest degree.

a) \[ \tan x = \frac{14}{8} \]
   \[ x = 60^\circ \]

b) \[ \cos x = \frac{24}{30} \]
   \[ x = 37^\circ \]

14. State the quadrant in which the terminal side of \( \theta \) lies if \( \tan \theta < 0 \) and \( \cos \theta > 0 \).

\[ \tan \theta = \frac{3}{4} \]

15. If \( \sec \theta = \frac{2\sqrt{3}}{3} \) and \( \sin \theta = -\frac{1}{2} \), find the exact value of \( \cot \theta = \frac{\cos \theta}{\sin \theta} \).

\[ \cot \theta = \frac{-\sqrt{3}}{2} \]

16. State two (positive and negative) coterminal angles for each of the following.

a) \[ \frac{2\pi}{3} = 120^\circ \]
   \[-240^\circ \rightarrow -\frac{4\pi}{3} \]
   \[+480^\circ \rightarrow \frac{8\pi}{3} \]

b) \[ \frac{11\pi}{6} = 330^\circ \]
   \[+390^\circ \rightarrow \frac{13\pi}{6} \]
   \[-30^\circ \rightarrow -\frac{\pi}{6} \]
   \[+225^\circ \rightarrow \frac{5\pi}{4} \]

17. Find each function value if \( \theta \) is a positive acute angle.

a) \[ \csc \theta \text{ if } \sin \theta = \frac{3}{\sqrt{5}} \]
   \[ \sin \theta = \frac{1}{\sqrt{5}} \]
   \[ \csc \theta = \frac{\sqrt{5}}{3} \]

b) \[ \cot \theta \text{ if } \sin \theta = \frac{3}{7} \]
   \[ \tan \theta = \frac{3}{4} \]
   \[ \cot \theta = \frac{4}{3} \]
18. Find each of the following. Use exact values only.

a) \( \sin \frac{7\pi}{4} = -\frac{\sqrt{2}}{2} \)

b) \( \sec \frac{2\pi}{3} = \frac{2}{1} \)

c) \( \cot \frac{11\pi}{6} = \frac{\sqrt{3}}{3} \)

d) \( \cos \frac{3\pi}{4} = -\frac{\sqrt{2}}{2} \)

19. Complete each of the following tables please.

<table>
<thead>
<tr>
<th>Radian Measure</th>
<th>( \frac{11\pi}{3} )</th>
<th>( \frac{11\pi}{6} )</th>
<th>( \frac{5\pi}{4} )</th>
<th>( \frac{5\pi}{2} )</th>
<th>( \frac{19\pi}{6} )</th>
<th>( -\frac{\pi}{4} )</th>
<th>( 3\pi )</th>
<th>( -\frac{7\pi}{6} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Degree Measure</td>
<td>330°</td>
<td>300°</td>
<td>225°</td>
<td>135°</td>
<td>45°</td>
<td>-45°</td>
<td>540°</td>
<td>-210°</td>
</tr>
<tr>
<td>( \sin \theta )</td>
<td>-\frac{\sqrt{3}}{2}</td>
<td>-\frac{1}{2}</td>
<td>-\frac{\sqrt{2}}{2}</td>
<td>1</td>
<td>-\frac{\sqrt{2}}{2}</td>
<td>0</td>
<td>\frac{1}{2}</td>
<td>0</td>
</tr>
<tr>
<td>( \cos \theta )</td>
<td>-\frac{1}{2}</td>
<td>\frac{\sqrt{3}}{2}</td>
<td>-\frac{\sqrt{2}}{2}</td>
<td>0</td>
<td>-\frac{\sqrt{2}}{2}</td>
<td>-1</td>
<td>-\frac{\sqrt{2}}{2}</td>
<td>0</td>
</tr>
<tr>
<td>( \tan \theta )</td>
<td>-\sqrt{3}</td>
<td>-\frac{\sqrt{3}}{3}</td>
<td>1</td>
<td>UNDF</td>
<td>\frac{\sqrt{3}}{3}</td>
<td>-1</td>
<td>0</td>
<td>-\frac{\sqrt{3}}{3}</td>
</tr>
</tbody>
</table>

20. During its approach to Earth, the space shuttle's glide angle changes. When the shuttle's altitude is about 15.7 miles, its horizontal distance to the runway is about 59 miles. What is its glide angle? Round your answer to the nearest tenth.

\[ \tan \theta = \frac{15.7}{59} \]

(2nd Tan⁻¹)

\[ \theta = 14.9° \]
21. Evaluate and find the exact value:

<table>
<thead>
<tr>
<th></th>
<th>( \sin \left( \frac{-\pi}{6} \right) )</th>
<th>( \tan \frac{5\pi}{4} )</th>
<th>( \tan \frac{11\pi}{6} )</th>
<th>( \sin \frac{7\pi}{4} )</th>
<th>( \cos \left( -\frac{\pi}{3} \right) )</th>
<th>( \cos \frac{7\pi}{3} = \csc \left( 420^\circ \right) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>a) ( \sin (-30^\circ) )</td>
<td>( -\frac{1}{2} )</td>
<td></td>
<td></td>
<td>( \sqrt{2} )</td>
<td>( \frac{1}{2} )</td>
<td>( \sqrt{2} )</td>
</tr>
<tr>
<td>b) ( \tan 225^\circ )</td>
<td>( 1 )</td>
<td></td>
<td></td>
<td>( \sqrt{2} )</td>
<td>( \frac{1}{2} )</td>
<td>( \sqrt{2} )</td>
</tr>
<tr>
<td>c) ( \tan 330^\circ )</td>
<td>( -\frac{\sqrt{3}}{3} )</td>
<td></td>
<td></td>
<td>( \sqrt{2} )</td>
<td>( \frac{1}{2} )</td>
<td>( \sqrt{2} )</td>
</tr>
</tbody>
</table>

22. Convert the following from degrees to radians or vice versa:

<table>
<thead>
<tr>
<th></th>
<th>( 36^\circ \left( \frac{\pi}{5} \right) )</th>
<th>( 17\pi / 20 )</th>
<th>( 195^\circ \left( \frac{7\pi}{8} \right) )</th>
<th>( 320^\circ \left( \frac{7\pi}{8} \right) )</th>
<th>( 12^\circ \left( \frac{\pi}{15} \right) )</th>
<th>( 300^\circ \left( \frac{5\pi}{3} \right) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>a) ( 36^\circ )</td>
<td>( \frac{\pi}{5} )</td>
<td></td>
<td></td>
<td>( \frac{7\pi}{8} )</td>
<td>( \frac{\pi}{15} )</td>
<td>( \frac{5\pi}{3} )</td>
</tr>
<tr>
<td>b) ( 17\pi / 20 )</td>
<td>( 153^\circ )</td>
<td></td>
<td></td>
<td>( 320^\circ )</td>
<td>( 12^\circ )</td>
<td>( 300^\circ )</td>
</tr>
<tr>
<td>c) ( 195^\circ )</td>
<td>( \frac{13\pi}{12} )</td>
<td></td>
<td></td>
<td>( \frac{7\pi}{8} )</td>
<td>( \frac{\pi}{15} )</td>
<td>( \frac{5\pi}{3} )</td>
</tr>
<tr>
<td>d) ( 320^\circ )</td>
<td></td>
<td></td>
<td></td>
<td>( \frac{7\pi}{8} )</td>
<td>( \frac{\pi}{15} )</td>
<td>( \frac{5\pi}{3} )</td>
</tr>
<tr>
<td>e) ( 12^\circ )</td>
<td></td>
<td></td>
<td></td>
<td>( \frac{\pi}{15} )</td>
<td>( \frac{\pi}{15} )</td>
<td>( \frac{5\pi}{3} )</td>
</tr>
<tr>
<td>f) ( 300^\circ )</td>
<td></td>
<td></td>
<td></td>
<td>( \frac{5\pi}{3} )</td>
<td>( \frac{5\pi}{3} )</td>
<td>( \frac{5\pi}{3} )</td>
</tr>
</tbody>
</table>

23. Find the values of the six trigonometric functions of \( \theta \), if \( \theta \) is an angle in standard position with the point \((-5, -12)\) on its terminal ray.

\[
\sin \theta = \frac{\text{opp}}{\text{hyp}} = -\frac{12}{13} \\
\cos \theta = \frac{\text{adj}}{\text{hyp}} = -\frac{5}{13} \\
\tan \theta = \frac{\text{opp}}{\text{adj}} = \frac{-12}{-5} = \frac{12}{5} \\
\csc \theta = \frac{1}{\sin \theta} = -\frac{13}{12} \\
\sec \theta = \frac{1}{\cos \theta} = -\frac{13}{5} \\
\cot \theta = \frac{1}{\tan \theta} = \frac{5}{12}
\]
24. Graph each of the following on the grid provided if $0 \leq x < 2\pi$ and state the range for each.

$$y = 2\sin(x) + 2$$

Range: $[0, 4]$  

25. The position of a particle at time, $t$, seconds is given by the function $y = -\sin(2t) + 3$, for $t > 0$.

a) Find the maximum and minimum values of $y$ and the times when they first occur.

$$\text{Max} = \text{Mid} + \text{AMP} = 3 + 1 = 4$$

$$\text{Min} = \text{Mid} - \text{AMP} = 3 - 1 = 2$$

b) Determine the period of the particle's movement.

$$P = \frac{2\pi}{2} = \pi$$

$c) Graph one cycle of$y$and state the amplitude, frequency, period, and midline.$

Amplitude = 1  
Midline = 3  
Frequency = 2  
Period = $\pi$
26. As a wave passes a jetty pylon, the depth of water, $D$, meters, at time, $t$, seconds is given by the function

$$D = 0.2 \sin\left(\frac{\pi}{4} t\right) + 5, t > 0.$$ 

Find the maximum and minimum depth of the water. When do these occur? How do you know? Explain.

\[ \text{Max} = 5 + 0.2 = 5.2 \quad \text{occurs at} \quad x = 2 \]

\[ \text{Min} = 5 - 0.2 = 4.8 \quad \text{occurs at} \quad x = 6 \]

27. The elk population in Denali National Park dropped from a high of 150,000 in 1950 to a low of 60,000 in 1975. However the population then rose back to a maximum of 150,000 again in 2000. Scientists hypothesize that this pattern will continue and that the elk population will fluctuate in this manner every 50 years.

a) Letting $t = 0$ in 1950, sketch the graph of this elk population from 1950 through 2000.

b) List the following information about the equation above:

\[ \text{AMP} = \frac{45,000}{20} = 105,000 \]

\[ \text{FREQ} = \frac{2\pi}{50} = \frac{\pi}{25} \]

\[ \text{PERIOD} = 50 \text{ yrs} \]

\[ y = 45,000 \cos\left(\frac{\pi}{25} x\right) + 105,000 \]

c) Using your equation, what would you estimate the elk population to be in 2020?

(Round your answer to the nearest whole elk!)

\[ 45,000 \cos\left(\frac{\pi}{25} (70)\right) + 105,000 = 68,544 \]

d) Rewrite this equation as a sine function with a horizontal shift.

\[ y = -45,000 \sin\left(\frac{\pi}{25} (x - 12.5)\right) + 105,000 \]