RATIONAL EXPONENTS AND RADICAL FUNCTIONS

Basic Exponents Properties

<table>
<thead>
<tr>
<th>Expression</th>
<th>Result</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x^a \cdot x^b = x^{a+b}$</td>
<td>$2x^5$</td>
</tr>
<tr>
<td>$\frac{x^a}{x^b} = x^{a-b}$</td>
<td>$\frac{5}{2}$</td>
</tr>
<tr>
<td>$(x^a)^b = x^{ab}$</td>
<td>$2^3$</td>
</tr>
<tr>
<td>$x^{m/n} = \sqrt[n]{x^m}$</td>
<td>$(2^2)^3$</td>
</tr>
</tbody>
</table>

(a) $x^3 \cdot x^{12}$
(b) $4x^3 \cdot 5x^5$
(c) $(-3x^2y)(5x^7y^3)$
(d) $(4x^3y^6)(-7x^4)$

(e) $\frac{x^9}{x^7}$
(f) $\frac{5x^3y^7}{15xy^2}$
(g) $\frac{x^3}{x^{10}}$
(h) $\frac{10x^4y^3}{25x^8}$

(i) $(x^5)^8$
(j) $(10x^3)^0$
(k) $(-4x^3)^3$
(l) $\frac{x^2y^4}{x^6y}$

Exercise: Rewrite each of the following power/root combinations as a rational exponent in simplest form.

(a) $\sqrt[3]{x^7}$
(b) $\sqrt[4]{x^6}$
(c) $(\sqrt{x})^6$
(d) $(\sqrt[10]{x})^3$

Simplifying Square Roots

Exercise #6: Simplify each of the following square roots. Show the manipulations that lead to your answers.

(a) $\sqrt{18x^4}$
(b) $\sqrt{200x^5y^3}$
(c) $\sqrt{147x^9y^5}$

Simplifying Higher Order Roots

Simplify each of the following higher order roots.

(a) $\sqrt[3]{16}$
(b) $\sqrt[10]{108}$
(c) $\sqrt[6]{162}$
(d) $\sqrt[4]{64x^{12}y^{15}}$
The Square Root Function
Consider the two functions \( f(x) = \sqrt{x} \) and \( g(x) = \sqrt{x+3} - 2 \). (All even powered roots will behave similarly.)

Explain how \( g(x) \) is transformed from \( f(x) \).

State the domain and range of each function
\[
\begin{align*}
f(x) &= \sqrt{x} \\
g(x) &= \sqrt{x+3} - 2
\end{align*}
\]

Domain: \( \{x \mid x \geq 0\} \)  
Domain: \( \{x \mid x \geq -3\} \)

Range: \( \{y \mid y \geq 0\} \)  
Range: \( \{y \mid y \geq -2\} \)

Determine the domains of each of the following functions.

(a) \( y = \sqrt{x+10} \)

\[
\begin{align*}
x + 10 &\geq 0 \\
x &\geq -10 \\
\{x \mid x \geq -10\}
\end{align*}
\]

(b) \( y = \sqrt{x^2 - 4x - 5} \)

\[
\begin{align*}
x^2 - 4x - 5 &\geq 0 \\
(x-5)(x+1) &\geq 0 \\
x &\geq 5 \text{ or } x = -1 \\
x = -2 &\Rightarrow (-2)^2 - 4(-2) - 5 = 7 \geq 0 \text{ Yes} \\
x = 0 &\Rightarrow (0)^2 - 4(0) - 5 = -5 \leq 0 \text{ No} \\
x = 6 &\Rightarrow (6)^2 - 4(6) - 5 = 7 \geq 0 \text{ Yes} \\
\{x \mid x \geq 5 \text{ or } x \geq -1\}
\end{align*}
\]

Solving Radical Equations
1. Isolate the radical
2. Raise both sides to the power of the index (or root value) to cancel the radical.

Exercise: Consider the system of equations shown below.

\[
y = \sqrt{x+3} \quad \text{and} \quad y = x + 1
\]

(a) Solve this system graphically using the grid to the right.

(b) Solve this system algebraically for only the \( x \)-values using substitution below.

\[
\begin{align*}
\sqrt{x+3} &= x + 1 \\
(x+3)^2 &= (x+1)^2 = (x+1)(x+1) \\
x + 3 &= x^2 + 2x + 1 \\
0 &= x^2 + x - 2 \\
0 &= (x+2)(x-1) \\
x &= -2 \text{ or } x = 1
\end{align*}
\]

Be careful to check your answers algebraically and reject any extraneous (false) roots!
Solving Radical Inequalities

Even roots:
1. Solve the radical equation for one of the bounds.
2. Find the domain restriction for the other bound.

Odd roots:
1. Solve the radical equation for one of the bounds.
2. There is no domain restriction for the other bound.

Exercise:
\[4\sqrt{x} + 3 \leq 23\]
\[\sqrt{x} + 7 \geq 3\]

CW-Inverses of Functions

Algebraically:
Switch the \(x\) and the \(y\) variables and then re-solve to get back into \(y = \) form.

Example:
\[y = \frac{2x - 4}{x + 5}\]

Graphically:
An inverse will be a reflection over the line \(y = x\) and will have diagonal line symmetry.

Example: \(y = 2x - 4\)  
(Note: switch the table of values to create the inverse function.)
UNIT 6: EXPONENTIAL AND LOGARITHMIC FUNCTIONS

Basic Exponential Function

\[ y = b^x \] where \( b > 0 \) and \( b \neq 1 \)

**Ex.** When \( b > 1 \), then the function is INCREASING

\[ y = 2^x \]

<table>
<thead>
<tr>
<th>( x )</th>
<th>( y = 2^x )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-3</td>
<td>( 2^{-3} = \frac{1}{2^3} = \frac{1}{8} )</td>
</tr>
<tr>
<td>-2</td>
<td>( 2^{-2} = \frac{1}{2^2} = \frac{1}{4} )</td>
</tr>
<tr>
<td>-1</td>
<td>( 2^{-1} = \frac{1}{2} )</td>
</tr>
<tr>
<td>0</td>
<td>( 2^0 = 1 )</td>
</tr>
<tr>
<td>1</td>
<td>( 2^1 = 2 )</td>
</tr>
<tr>
<td>2</td>
<td>( 2^2 = 4 )</td>
</tr>
<tr>
<td>3</td>
<td>( 2^3 = 8 )</td>
</tr>
</tbody>
</table>

The growth factor \( b \) can be rewritten as either \( 1 + r \) or \( 1 - r \), depending whether there is exponential growth or decay. (\( r \) is the percent rate – always written in decimal form!)

\[ f(x) = a \cdot b^x \] can be written as \( f(x) = a \cdot (1 + r)^x \) \( \Rightarrow \) will be exponential growth.

\[ f(x) = a \cdot b^x \] can be written as \( f(x) = a \cdot (1 - r)^x \) \( \Rightarrow \) will be exponential decay.

**Example:** Write a function that takes an initial value of 4 and increases it by 20\% every time \( x \) increases by 1 (for every unit increase in \( x \)).

**Solution:** We let our initial value be the \( a \), so \( a = 4 \). Our growth rate is 20\%, so \( r = 0.20 \).

Our function must be \( f(x) = 4(1.20)^x \)

**Transformations of Exponential Functions**

Parent function for all exponentials: \( f(x) = a \cdot b^x \) The sketches below are for \( a > 0 \).

This parent function is **asymptotic** to the \( x \)-axis (the line whose equation is \( y = 0 \)).

It must pass through the point \((0, a)\), and it has a **domain** of \((-\infty, \infty)\) and a **range** of \((0, \infty)\).

<table>
<thead>
<tr>
<th>( x )</th>
<th>( y = \left(\frac{1}{2}\right)^x )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-3</td>
<td>8</td>
</tr>
<tr>
<td>-2</td>
<td>4</td>
</tr>
<tr>
<td>-1</td>
<td>2</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>0.5</td>
</tr>
<tr>
<td>2</td>
<td>0.25</td>
</tr>
<tr>
<td>3</td>
<td>0.125</td>
</tr>
</tbody>
</table>

\[ f(x) = a \cdot b^x + k \] is shifted **UP** \( k \) units.

\[ f(x) = a \cdot b^x - k \] is shifted **DOWN** \( k \) units.

\[ f(x) = a \cdot b^x - k \] is shifted **RIGHT** \( k \) units.

\[ f(x) = a \cdot b^x + k \] is shifted **LEFT** \( k \) units.
Finding an Equation of an Exponential Function

Write an exponential function that models the data in the table:

<table>
<thead>
<tr>
<th>$x$</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y$</td>
<td>300</td>
<td>150</td>
<td>75</td>
<td>37.5</td>
<td>18.75</td>
<td>9.375</td>
</tr>
</tbody>
</table>

When given a set of data, you can check the ratio of the outputs to determine if the data is exponential. If the ratios are consistent, then it is considered exponential.

**To find an equation using a regression:**
1. Type data into L1 and L2 (hit STAT → EDIT → ENTER)
2. Use the calculator to generate the regression equation: (hit STAT → CALC → #0 EXP REG)
3. Copy the info from the calculator: $y = a \cdot b^x$, $a = \_\_\_\_$, $b = \_\_\_\_$.

**To find an equation algebraically:**

**Example #1**
An exponential function of the form $y = a(b)^x$ passes through the points $(2, 36)$ and $(5, 121.5)$.

(a) By substituting these two points into the general form of the exponential, create a system of equations in the constants $a$ and $b$.

\[
\begin{align*}
36 &= a(b)^2 \\
121.5 &= a(b)^5
\end{align*}
\]

(b) Divide these two equations to eliminate the constant $a$. Recall that when dividing to like bases, you subtract their exponents.

\[
\frac{121.5}{36} = \frac{a(b)^5}{a(b)^2} = b^3
\]

(c) Solve the resulting equation from (b) for the base, $b$.

\[
b^3 = 3.375 \rightarrow b = \left(\frac{3.375}{3.375}\right)^{\frac{1}{3}} = \sqrt[3]{3.375} = b = 1.5
\]

(d) Use your value from (c) to determine the value of $a$. State the final equation.

\[
36 = a(1.5)^2 \rightarrow a = \frac{36}{1.5^2} = 16 \rightarrow y = 16(1.5)^x
\]

**Example #2**
A bacterial colony is growing at an exponential rate. It is known that after 4 hours, its population is at 98 bacteria and after 9 hours it is 189 bacteria. Determine an equation in $y = a(b)^x$ form that models the population, $y$, as a function of the number of hours, $x$. At what percent rate is the population growing per hour?

\[
\begin{align*}
\frac{a(b)^9}{a(b)^4} &= \frac{189}{98} \\
\frac{b^5}{b^4} &= \frac{1.92857...}{b} \\
b &= \sqrt[5]{1.92857...} = 1.1403... \\
\frac{98}{a(1.1403...)^4} &= 189 \\
a &= \frac{98}{1.1403...} = 85.79478... \approx 58 \\
y &= 58(1.14)^x
\end{align*}
\]

Because our base value is 1.14, the population of bacteria is growing at a rate of 14% per hour.
Modeling with Different Forms of Exponential Functions

<table>
<thead>
<tr>
<th>Annual Growth:</th>
<th>Annual Decay:</th>
<th>Compound Interest:</th>
<th>Continuous Compounding of Interest:</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y = ab^t )</td>
<td>( y = ab^t )</td>
<td>( y = a(1+r)^t )</td>
<td>( y = ae^{kt} )</td>
</tr>
<tr>
<td>( y = a(1+r)^t )</td>
<td>( y = a(1-r)^t )</td>
<td>( y = a\left(1+\frac{r}{n}\right)^{nt} )</td>
<td></td>
</tr>
</tbody>
</table>

**Example #1:**
A warm glass of water, initially at 120 degrees Fahrenheit, is placed in a refrigerator at 34 degrees Fahrenheit and its temperature is seen to decrease according to the exponential function

\[ T(h) = 86(0.83)^h + 34 \]

(a) Verify that the temperature starts at 120 degrees Fahrenheit by evaluating \( T(0) \).

\[ T(0) = 86(0.83)^0 + 34 = 86(1) + 34 = 120 \]

(b) Using your calculator, sketch a graph of \( T \) below for all values of \( h \) on the interval \( 0 \leq h \leq 24 \). Be sure to label your y-axis and y-intercept.

(c) After how many hours will the temperature be at 50 degrees Fahrenheit? State your answer to the nearest hundredth of an hour. Illustrate your answer on the graph you drew in (b).

\[ 86(0.83)^t + 34 = 50 \]
\[ t = 9.03 \text{ hrs} \]

**Example #2:**
The value of an initial investment of $400 at 3% nominal interest compounded quarterly can be modeled using which of the following equations, where \( t \) is the number of years since the investment was made?

\( (1) \ A = 400(1.0075)^{4t} \)
\( (2) \ A = 400(1.0075)^{t} \)
\( (3) \ A = 400(1.03)^{4t} \)
\( (4) \ A = 400(1.0303)^{3t} \)

\[ A = P\left(1+\frac{r}{n}\right)^{nt} = 400\left(1+\frac{0.03}{4}\right)^{4t} = 400(1.0075)^{4t} \]

(1)

**Example #3:**
Franco invests $4,500 in an account that earns a 3.8% nominal interest rate compounded continuously. If he withdraws the profit from the investment after 5 years, how much has he earned on his investment?

\( (1) \ $858.92 \)
\( (2) \ $912.59 \)
\( (3) \ $922.50 \)
\( (4) \ $941.62 \)

\[ A(t) = Pe^{rt} = 4500e^{0.038t} \]
\[ A(5) = 4500e^{0.19} \approx 5441.62 \]
\[ 5441.62 - 4500 = 941.62 \]

(4)
Solving Exponential Equations Using Common Bases
Solve each of the following equations by finding a common base for each side.

(a) \(8^x = 32\)

\[
\begin{align*}
(2^3)^x &= 2^x \\
2^x &= 2^5 \\
x &= \frac{5}{3}
\end{align*}
\]

(b) \(9^{2x+1} = 27\)

\[
\begin{align*}
(3^2)^{2x+1} &= 3^3 \\
3^{4x+2} &= 3^3 \\
4x + 2 &= 3 \\
x &= \frac{1}{4}
\end{align*}
\]

(c) \(125^x = \left(\frac{1}{25}\right)^{4-x}\)

\[
\begin{align*}
(5^3)^x &= (5^{-2})^{4-x} \\
5^{3x} &= 5^{-8+2x} \\
3x &= -8 + 2x \\
x &= -8
\end{align*}
\]

Introduction to the Logarithm Function
Defining Logarithmic Functions – The function \(y = \log_b x\) is the name we give the inverse of \(y = b^x\). For example, \(y = \log_2 x\) is the inverse of \(y = 2^x\). We can write an equivalent exponential equation for each logarithm as follows:

\[y = \log_b x\] is the same as \[b^y = x\]

The function \(f(x) = 2^x\)

<table>
<thead>
<tr>
<th>(x)</th>
<th>-3</th>
<th>-2</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>(f(x) = 2^x)</td>
<td>(\frac{1}{8})</td>
<td>(\frac{1}{4})</td>
<td>(\frac{1}{2})</td>
<td>1</td>
<td>2</td>
<td>4</td>
<td>8</td>
</tr>
</tbody>
</table>

The inverse function \(f^{-1}(x) = \log_2(x)\)

<table>
<thead>
<tr>
<th>(x)</th>
<th>(\frac{1}{8})</th>
<th>(\frac{1}{4})</th>
<th>(\frac{1}{2})</th>
<th>1</th>
<th>2</th>
<th>4</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>(f^{-1}(x))</td>
<td>-3</td>
<td>-2</td>
<td>-1</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
</tbody>
</table>

Notice that, as always, the graphs of \(f(x)\) and \(f^{-1}(x)\) are symmetric across \(y = x\).

Solving Exponential Equations Using Logarithms

Example #1:

Solve each of the following equations for the value of \(x\). Round your answers to the nearest hundredth.

(a) \(5^x = 18\)

\[
\begin{align*}
\log(5^x) &= \log 18 \\
x \log 5 &= \log 18 \Rightarrow x &= \frac{\log 18}{\log 5} \approx 1.80
\end{align*}
\]

(b) \(4(2)^x - 3 = 17\)

\[
\begin{align*}
4(2)^x - 3 &= 17 \\
4(2)^x &= 20 \\
(2)^x &= 5 \\
x &= \log_2(5)
\end{align*}
\]