RATIONAL EXPONENTS AND RADICAL FUNCTIONS

Basic Exponent Properties

<table>
<thead>
<tr>
<th>EXPONENT LAWS</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. ( x^a \cdot x^b = x^{a+b} )</td>
</tr>
<tr>
<td>5. ( x^{-a} = \frac{1}{x^a} ) and ( \frac{1}{x^{-a}} = x^a )</td>
</tr>
</tbody>
</table>

(a) \( x^3 \cdot x^{12} \)  
(b) \( 4x^3 \cdot 5x^5 \)  
(c) \( (-3x^2y)(5x^7y^3) \)  
(d) \( (4x^3y^6)(-7x^4) \)

(e) \( \frac{x^9}{x^3} \)  
(f) \( \frac{5x^3y^7}{15xy^3} \)  
(g) \( \frac{x^3}{x^{10}} \)  
(h) \( \frac{10x^4y^3}{25x^8} \)

(i) \( (x^3)^8 \)  
(j) \( (10x^3)^0 \)  
(k) \( (-4x^3)^3 \)  
(l) \( \frac{x^2y^4}{x^5y} \)

Exercise: Rewrite each of the following power/root combinations as a rational exponent in simplest form.

(a) \( \sqrt{x} \)  
(b) \( \sqrt[6]{x} \)  
(c) \( \sqrt[6]{x} \)  
(d) \( \sqrt[10]{x} \)

Simplifying Square Roots

Exercise #6: Simplify each of the following square roots. Show the manipulations that lead to your answers.

(a) \( \sqrt{18x^4} \)  
(b) \( \sqrt{200x^2y^3} \)  
(c) \( \sqrt{147x^6y^4} \)

Simplifying Higher Order Roots

Simplify each of the following higher order roots.

(a) \( \sqrt[3]{16} \)  
(b) \( \sqrt[6]{108} \)  
(c) \( \sqrt[8]{162} \)  
(d) \( \sqrt[15]{64x^{12}y^{17}} \)
The Square Root Function
Consider the two functions $f(x) = \sqrt{x}$ and $g(x) = \sqrt{x+3} - 2$.
(All even powered roots will behave similarly.)

Explain how $g(x)$ is transformed from $f(x)$.
$g(x)$ is formed by shifting $f(x)$ to the left 3 and down 2.

State the domain and range of each function
\[ f(x) = \sqrt{x} \quad g(x) = \sqrt{x+3} - 2 \]

Domain: \( \{x | x \geq 0\} \)  \quad Domain: \( \{x | x \geq -3\} \)

Range: \( \{y | y \geq 0\} \)  \quad Range: \( \{y | y \geq -2\} \)

Determine the domains of each of the following functions.

(a) \( y = \sqrt{x+10} \quad \text{Must be} \geq 0 \) \quad (b) \( y = \sqrt{x^2 - 4x - 5} \quad \text{Must be} \geq 0 \)

\[ x+10 \geq 0 \]
\[ x \geq -10 \]
\[ \{x | x \geq -10\} \]

\[ x^2 - 4x - 5 \geq 0 \]
\[ (x-5)(x+1) = 0 \]
\[ x = 5 \text{ or } x = -1 \]
\[ x = -2 \Rightarrow (-2)^2 - 4(-2) - 5 = 7 \geq 0 \text{ Yes} \]
\[ x = 0 \Rightarrow (0)^2 - 4(0) - 5 = -5 \leq 0 \text{ No} \]
\[ x = 6 \Rightarrow (6)^2 - 4(6) - 5 = 7 \geq 0 \text{ Yes} \]
\[ \{x | x \leq -1 \text{ or } x \geq 5\} \]

Solving Radical Equations
1. Isolate the radical
2. Raise both sides to the power of the index (or root value) to cancel the radical.

Exercise: Consider the system of equations shown below.
\[
y = \sqrt{x+3} \quad \text{and} \quad y = x+1
\]

(a) Solve this system **graphically** using the grid to the right.
(b) Solve this system **algebraically** for only the x-values using substitution below.

\[
\sqrt{x+3} = x+1
\]
\[
(x+1)^2 = (x+1)(x+1)
\]
\[
x+3 = x^2 + 2x + 1
\]
\[
x = x^2 + x - 2
\]
\[
0 = (x+2)(x-1)
\]
\[
x = -2 \text{ or } x = 1
\]

Be careful to check your answers algebraically and reject any extraneous (false) roots!
Solving Radical Inequalities

Even roots:
1. Solve the radical equation for one of the bounds.
2. Find the domain restriction for the other bound.

Odd roots:
1. Solve the radical equation for one of the bounds.
2. There is no domain restriction for the other bound.

Exercise:
\[4 \sqrt{x} + 3 \leq 23\]
\[-4 \leq x \leq 25\]
\[\left(\sqrt{5x}\right)^3 \leq 3(3)^3\]
\[x \geq 20\] or \([20, \infty)\]

CW-Inverses of Functions

Algebraically:
Switch the x and y variables and then re-solve to get back into \(y = \) form.

Example: \(y = 2x - 4\)
\[x = 2y - 4\]
\[x + 4 = 2y\]
\[y = \frac{x + 4}{2}\]

Graphically:
An inverse will be a reflection over the line \(y = x\) and will have diagonal line symmetry.

Example: \(y = 2x - 4\)
(Now switch the table of values to create the inverse function.)

<table>
<thead>
<tr>
<th>x</th>
<th>y</th>
</tr>
</thead>
<tbody>
<tr>
<td>-2</td>
<td>-8</td>
</tr>
<tr>
<td>-1</td>
<td>-6</td>
</tr>
<tr>
<td>0</td>
<td>-4</td>
</tr>
<tr>
<td>1</td>
<td>-2</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>2</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>x</th>
<th>y</th>
</tr>
</thead>
<tbody>
<tr>
<td>-8</td>
<td>-2</td>
</tr>
<tr>
<td>-6</td>
<td>-1</td>
</tr>
<tr>
<td>-2</td>
<td>0</td>
</tr>
<tr>
<td>-2</td>
<td>1</td>
</tr>
<tr>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
</tr>
</tbody>
</table>
Find the real solution(s) of the equation. Round your answer to two decimal places.

| 1. \( \frac{2}{3}x^3 = \frac{3072}{3} \) | 2. \( \sqrt[3]{x} = \sqrt[3]{1024} \) |
| \( x = 4 \) | \( x + \frac{3}{2} = \sqrt[3]{50} \) |
| \( x = \frac{\sqrt[3]{50}}{2} - \frac{3}{2} \) |

Simplify the expression.

| 3. \( \frac{\sqrt[6]{729}}{9^{\frac{1}{3}}} \) | 4. \( \sqrt[3]{500} \) |
| \( \left( \frac{8^\frac{1}{3}}{3} \right)^3 = \frac{\sqrt[3]{500}}{\sqrt[3]{\sqrt[3]{125}}} = \frac{5\sqrt[3]{4}}{5} \) |
| 8 |

| 5. \( \sqrt[3]{32x^6y^8z^4} \) | 6. \( 8^{\frac{4}{3}} \) |
| \( 2yz \sqrt[3]{xy} \) | \( \sqrt[4]{8^4} = 2^4 = 16 \) |

| 7. \( \frac{\sqrt{208}}{\sqrt{13}} = \sqrt[2]{16} = 2 \) | 8. \( (-125)^{\frac{2}{3}} \) |
| \( \sqrt[3]{-125} = (-5)^2 = 25 \) |

| 9. \( 12\sqrt[5]{5} - 2\sqrt[5]{125} \) | 10. \( \sqrt[3]{81} \) |
| \( 12\sqrt[5]{5} - 2(5\sqrt[5]{5}) = 12\sqrt[5]{5} - 10\sqrt[5]{5} \) | \( \frac{\sqrt[3]{3}}{\sqrt[3]{x^3}} \) |
| \( \sqrt[3]{5} \) |

| 11. \( \sqrt[3]{6n} - 26^{\frac{1}{2}}n^2 \) | 12. \( \sqrt[3]{625m^{10}} \) |
| \( 7b\sqrt[5]{n} - 2b^2\sqrt[5]{n^2} \) | \( \sqrt[5]{(m^2)(m^2)(m^2)(m^2)} \) |
| \( 5m^2 \sqrt[5]{n^2} \) |
20. A cylindrical container of water has a volume of 190 cubic inches. The radius \( r \) of the container can be found by using the formula \( r = \sqrt{\frac{V}{\pi h}} \), where \( V \) is the volume of the container and \( h \) is the height.

a) If the radius of the container is 3.5 inches, find the height. Round your answer to the nearest hundredth.

\[
(3.5)^2 = \left(\sqrt{\frac{190}{\pi h}}\right)^2 \Rightarrow 12.25 = \frac{190}{\pi h} \quad \frac{12.25 \pi h}{12.25} \quad h = 4.94
\]

b) If the height of the container is 10 inches, find the radius. Round your answer to the nearest hundredth.

\[
r = \sqrt{\frac{190}{\pi (10)}} = 2.459
\]

21. At the circus, the length of time \( t \) (in seconds) it takes for a trapeze artist to complete one full walk is given by the equation \( t = 2.31\ell^{2/3} \), where \( \ell \) is the length (in feet) of the trapeze line. The table below shows the length of the lines a certain performer must walk each show. How long will each walk take? Round your answers to the nearest tenth.

<table>
<thead>
<tr>
<th>Act</th>
<th>Walk length</th>
<th>Time</th>
</tr>
</thead>
<tbody>
<tr>
<td>Act 1</td>
<td>60 feet</td>
<td>17.9</td>
</tr>
<tr>
<td>Act 2</td>
<td>40 feet</td>
<td>14.6</td>
</tr>
<tr>
<td>Act 3</td>
<td>100 feet</td>
<td>23.1</td>
</tr>
<tr>
<td>Act 4</td>
<td>300 feet</td>
<td>40.0</td>
</tr>
</tbody>
</table>

22. Let \( f(x) = -2x^{2/3} \) and \( g(x) = -x^{2/3} \). Find \( (f + g)(x) \) and \( (f - g)(x) \) and state the domain of each.

Then evaluate \( (f + g)(243) \) and \( (f - g)(243) \).

\[
(f + g)(x) = -2x^{2/3} - x^{2/3} = -3x^{2/3} \quad (f + g)(243) = -3(243)^{2/3} = \frac{27}{-27} = 9
\]

\[
(f - g)(x) = -2x^{2/3} - (-x^{2/3}) = -x^{2/3} \quad (f - g)(243) = -(243)^{2/3} = -9
\]
23. Let \( f(x) = \frac{2}{3}x^2 \) and \( g(x) = -4x \). Find \((f \circ g)(x)\) and \(\left(\frac{f}{g}\right)(x)\) and state the domain of each. Then evaluate \((f \circ g)(4)\) and \(\left(\frac{f}{g}\right)(4)\).

\[
(f \circ g)(x) = \left(\frac{2}{3}x^2\right)(-4x) = -\frac{8}{3}x^3
\]

\[
(f \circ g)(4) = -\frac{8}{3}(4)^3 = -\frac{256}{3}
\]

\[
\left(\frac{f}{g}\right)(x) = \frac{\frac{2}{3}x^2}{-4x} = \frac{-\frac{1}{6}x}{x^2}
\]

\[
\left(\frac{f}{g}\right)(4) = \frac{-\frac{1}{6}(4)}{4^2} = \frac{-1}{3}
\]

Solve and check your answers.

24. \((4p - 7)^2 = 5^2\)

\[
p - \frac{7}{4} = \pm \frac{5}{1}
\]

\[
p = 8
\]

25. \(\sqrt{x+1} = -4\)

No solution! You must check your answers.

26. \((\sqrt{1-x})^2 = (-2)^2\)

\[
1 - x = -8
\]

\[
x = 9
\]

27. \(\sqrt{12 + x} = x\)

\[
x = 4
\]

28. \(y = \frac{y^2}{4} \quad (\text{where } y \neq 0)\)

\[
y^2 - 16y = 0
\]

\[
y(y - 16) = 0
\]

\[
(y = 0) \quad (y = 16)
\]

29. \(y = \sqrt{5y} \quad (\text{where } y \neq 0)\)

\[
y^2 = 25y
\]

\[
y^2 - 25y = 0
\]

\[
y(y - 25) = 0
\]

\[
(y = 0) \quad (y = 25)
\]

30. \((x^2 + 3\sqrt{x} - 2)^2\)

\[
x^2 > 9(x - 2)
\]

\[
x^2 > 9x - 18
\]

\[
x^2 - 9x + 18 > 0
\]

\[
(x - 6)(x - 3) > 0
\]

\[
x < 6 \quad x > 3
\]

31. \((\sqrt{1 - 3x} < x + 1)^2\)

\[
1 - 3x < x^2 + 2x + 1 + 3x - 1
\]

\[
x^2 + 5x > 0
\]

\[
x(x + 5) > 0
\]

\[
x < 0 \quad x > -5
\]

Check graph.

Graph does not exist here!
32. \( (\sqrt{x^2 - 2x - 5})^2 = (x+1)^2 \)
\[ x^2 - 2x - 5 = x^2 + 2x + 1 + \frac{2x}{1} \]
\[ x - 2x - 5 = x^2 + 2x + 1 + \frac{2x}{1} \]
\[ -7 = 4x \quad \therefore x = -\frac{7}{4} \quad \text{Check it!} \]
\[ \text{No solution!} \]

33. \( \sqrt{3x-5} = 7 - \sqrt{x+2} \)
\[ \text{Omit!} \]

34. \( 2(3y-4)^\frac{3}{2} - 4 = 50 \)
\[ \frac{3y-4}{2} = \frac{5y}{a} \]
\[ \left( \frac{3y-4}{2} \right)^3 = \left( \frac{5y}{a} \right)^3 \]
\[ \left( \frac{3y-4}{2} \right)^3 = \left( \frac{247}{3} \right)^3 \]
\[ \frac{y - 207}{3} = \frac{y}{2} \]
\[ \therefore y = \frac{207}{5} \]

35. \( (2y - \frac{5}{16})^\frac{3}{4} - \frac{1}{8} = 7 \)
\[ (2y - \frac{5}{16})^\frac{3}{4} = \frac{7}{1} + \frac{1}{8} \]
\[ (2y - \frac{5}{16})^\frac{3}{4} = \frac{63}{8} \]
\[ 2y - \frac{5}{16} = \frac{63}{8} \]
\[ \therefore y = \frac{27}{16} \]

Find the inverse of each.

36. \( f(x) = -2x^2 - 1 \)
\[ x = \sqrt{-y + 2} \quad y = \sqrt{x + 1} \]
\[ x + 2 = \sqrt{\frac{x + 1}{2}} \]

37. \( g(x) = -\frac{2}{x - 1} - 3 \)
\[ x = \frac{y}{y^3 - 1} \]
\[ x + 3 = \frac{-2}{y} \]

38. \( f(x) = -\sqrt[3]{x - 2} \)
\[ x = \frac{y - 3}{y - 1} \]
\[ x + 2 = \sqrt[3]{y - 1} \]

39. \( f(x) = \frac{-4}{7} x - \frac{16}{7} \)
\[ 7x = -4y + 16 \]
\[ x = \frac{-4y - 16}{7} \]
\[ \therefore y = 7x + 16 \]

40. \( g(x) = \frac{1}{-x + 3} - 1 \)
\[ x = \frac{y}{-y + 3} - 1 \]
\[ x + 1 = \frac{2y + 3}{y + 3} \]

41. \( f(x) = -\frac{10 - 5x}{2} \)
\[ x = \frac{-10 + 5y}{2} \]
\[ x + 10 = \frac{5y}{2} \]
\[ \therefore y = 2x + 10 \]

42. \( f(x) = -x - 2x^2 \)
\[ x = \frac{-1}{x + 3} - 1 \]
\[ \frac{x + 3}{2} = \frac{x}{y + 3} \]

43. \( h(x) = 3x^3 + 3 \)
\[ x = \frac{3y^3 - 3}{3} \]
\[ x - 3 = \frac{3y^3 - 3}{3} \]
<table>
<thead>
<tr>
<th>44. $2\sqrt{x} - 5 \geq 3$</th>
<th>45. $\sqrt{x-4} \leq 5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x \geq 16$ or $[16, \infty)$</td>
<td>$x \leq 29$ Check graph!</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>46. $2\sqrt{-(x-4)} - 6 \leq 0$</th>
<th>47. $-\sqrt{x+2} \geq -4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-5 \leq x \leq 4$</td>
<td>$-2 \leq x \leq 14$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>48. $4\sqrt{x} + 1 &lt; 9$</th>
<th>49. $-\sqrt{x-3} + 6 \leq 2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x &lt; 4$</td>
<td>$x \geq 19$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>50. $\sqrt{-x+3} &gt; 5$</th>
<th>51. $(\sqrt{x+7})^2 \geq 3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x &lt; -4$ or $(-\infty, -4)$</td>
<td>$x \geq 2$</td>
</tr>
</tbody>
</table>

Graphs shown for selected inequalities.