QUADRATIC FUNCTIONS

Any function of the form \( f(x) = ax^2 + bx + c \) where the leading coefficient, \( a \), is not zero.

a) Standard Form: \( f(x) = ax^2 + bx + c \) useful for finding the \( y \)-intercept (the "c" value)

b) Vertex Form: \( f(x) = a(x - h)^2 + k \) useful for finding the vertex \((h, k)\)

c) Factored Form: \( f(x) = a(x - r_1)(x - r_2) \) useful for finding the roots: \( x = r_1, x = r_2 \)

Definitions:

1. Turning Point (Vertex): the lowest (if face up) or highest (if face down) point of a parabola.
2. Axis of Symmetry: the line of symmetry down the middle of the parabola \( (x = -\frac{b}{2a}) \)
3. Focus: the coordinate that the parabola bends around, \( \frac{1}{4a} \) units away from vertex.
4. Directrix: the line that the parabola bends away from (usually horizontal line)
5. Roots: the \( x \)-intercepts of the parabola

Consider the simplest of all quadratic functions \( y = x^2 \).

(a) Create a table of values to plot this function over the domain interval \(-3 \leq x \leq 3\).

<table>
<thead>
<tr>
<th>( x )</th>
<th>( -3 )</th>
<th>( -2 )</th>
<th>( -1 )</th>
<th>( 0 )</th>
<th>( 1 )</th>
<th>( 2 )</th>
<th>( 3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y = x^2 )</td>
<td>9</td>
<td>4</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>4</td>
<td>9</td>
</tr>
</tbody>
</table>

(b) Sketch a graph of this function on the grid to the right.

(c) State the coordinates of the turning point of this parabola. \( (0, 0) \)

(d) State the equation of this parabola's axis of symmetry. \( x = 0 \)

(e) Over what interval is this function increasing? Decreasing? \( 0 < x < \infty \), \( -\infty < x < 0 \)

How is \( x = y^2 \) different? Sketch the graph to the right.

State the Domain: \( x \geq 0 \) or \( [0, \infty) \)

State the Range: \( \text{All Real Numbers} (-\infty, \infty) \)
Example: Use the formula \( x = \frac{-b}{2a} \) to find the turning points for each of the following quadratic functions.

(a) \( f(x) = 2x^2 - 12x + 7 \)

\[
x = \frac{-(-12)}{2(2)} = \frac{12}{4} = 3
\]

\[
y = 2(3)^2 - 12(3) + 7 = -11
\]

Turning point: \((3, -11)\)

(b) \( g(x) = -\frac{1}{4}x^2 + 5x - 20 \)

\[
x = \frac{-5}{2(-\frac{1}{4})} = \frac{-5}{-\frac{1}{2}} = 10
\]

\[
y = -\frac{1}{4}(10)^2 + 5(10) - 20 = 5
\]

Turning Point: \((10, 5)\)

("Always verify by graphing the function on your calculator!
"(Check on your table of values)"

Rewrite in Vertex form: \( y = \frac{1}{2}(x-3)^2 - 11 \)

Rewrite in Vertex form: \( y = -\frac{1}{4}(x-10)^2 + 5 \)

Describe the transformation from \( y = x^2 \):
- Shif\(t\)ed right 3 units, vertically stretched by a factor of 2, and shifted down 11 units.

Describe the transformation from \( y = x^2 \):
- Shif\(t\)ed right 10 units, vertically shrunk by a factor of \( \frac{1}{4} \), stretched up 5 units, and reflected over \( x\)-axis then.

Quadratic Regressions:

1. Hit STAT→EDIT. Enter data into L1, L2
2. Hit STAT→CALC. Select option #5 QuadReg
3. Scroll down to Calculate.
4. Copy down the equation. Round coefficients as specified.

Example:
A baseball is thrown up in the air. The table shows the heights \( y \) (in feet) of the baseball after \( x \) seconds.

<table>
<thead>
<tr>
<th>Time, ( x )</th>
<th>0</th>
<th>2</th>
<th>4</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Baseball height, ( y )</strong></td>
<td>6</td>
<td>22</td>
<td>22</td>
<td>6</td>
</tr>
</tbody>
</table>

\[
y = -2x^2 + 12x + 6
\]

(a) Write an equation for the path of the baseball.

(b) Find the height of the baseball after 5 seconds.

\( x = 5 \) check table, \( y = 16 \) ft

(c) What is the maximum height of the baseball?

2nd TRACE→MAXIMUM: \( x = 3, \ y = 24 \ ft \)

(d) When does the baseball reach its maximum height?

\( \text{time} = 3 \text{ sec} \)
Focus & Directrix of a Parabola

A parabola is the collection of all points equidistant from a fixed point (known as its focus) and a fixed line (known as its directrix).

Consider a parabola that is the collection of all points equidistant from the point \((0, 8)\) and the line \(y = 2\).

(a) Give each of the following:
- Directrix: \(y = 2\)
- Focus: \((0, 8)\)

(b) Draw a diagram of this parabola and label its turning point on the diagram below.

(c) Find the equation of this parabola.

(Use the formula \(a = \frac{1}{4p}\))

\[
y = a(x - 5)^2 + 0 \quad \text{so} \quad a = \frac{1}{4(5)} = \frac{1}{20}
\]

\[
\sqrt{y} = \frac{1}{10}(x - 5)^2
\]

Example 1:

Determine the equation of the parabola whose focus is the point \((4, 1)\) and whose directrix is the horizontal line \(y = -3\). First, draw a diagram that shows the parabola, then carefully use the formula \(a = \frac{1}{4p}\) to derive its equation.

\[
y = a(x - 4)^2 - 1
\]

\[
\sqrt{y} = \frac{1}{8}(x - 4)^2 - 1
\]

Example 2:

Given the equation of a parabola to be \(y = \frac{1}{12}x^2 + 1\), determine the following and sketch the graph labeling each of the parameters listed below:

a) Vertex \((0, 1)\)

b) "a" value \(\frac{1}{12}\) \(a = \frac{1}{4p}\) so \(\frac{1}{12} = \frac{1}{4p}\) so \(p = 3\)

c) "p" value 3

d) Focus \((0, 4)\)

e) Directrix \(y = -2\)
1. A parabola has an axis of symmetry \( x = -2 \) and passes through the point \((-5, 6)\). Find another point that lies on the graph of the parabola.

2. Let the graph of \( g \) be a vertical stretch by a factor of 4 and a reflection in the x-axis of the graph of \( f(x) = x^2 - 3 \). Write a rule for \( g \).

\[
g(x) = -4f(x)
\]

3. Identify the focus, directrix, and axis of symmetry of \( f(x) = \frac{1}{16}x^2 \).

4. Write the equation of the parabola.

   a) \[
y = a(x - v)^2 + h
\]

   b) \[
y = \frac{1}{4}(x - 2)^2 + 4
\]
5. A bridge builder plans to construct a cable suspension bridge in your town. The cable being used will form a curve modeled by the equation \( h(x) = 3x^2 - 6x + 200 \), where \( x \) represents the length of cable used (in feet) and \( h(x) \) represents the height of the cable (in feet). At what height will the cable hang closest to the bridge deck?

\[ (1, 197) \]

6. Graph \( f(x) = 3(x + 1)^2 - 3 \). Label the vertex and axis of symmetry. Describe where the function is increasing and decreasing.

\[ \sqrt{(-1) - 3} \]

Axis of Symmetry: \( x = -1 \)

Increasing on the right of \( x = -1 \)
Decreasing on the left of \( x = -1 \)

7. Graph \( g(x) = -2x^2 - 4x + 3 \). Label the vertex and axis of symmetry. Describe where the function is increasing and decreasing.

\[ \sqrt{(-1) - 5} \]

Axis of Symmetry: \( x = -1 \)

Increasing on the left of \( x = -1 \)
Decreasing on the right of \( x = -1 \)
8. The tables show the number of toy bears a toy manufacturer can sell.

<table>
<thead>
<tr>
<th>Price (dollars), x</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bears sold (in thousands), y</td>
<td>84</td>
<td>96</td>
<td>100</td>
<td>96</td>
<td>84</td>
</tr>
</tbody>
</table>

\[ y = ax^2 + bx + c \]
\[ a = -4 \quad b = 40 \quad c = 0 \]

\[ y = -4x^2 + 40x \]

a) Write a quadratic equation, rounding coefficients to the nearest hundredth.

\[ y = \text{ Bears sold (in thousands)} \]
\[ (-1, 84) \quad (0, 96) \quad (1, 100) \]

b) Using your equation found in part (a), determine how many bears the manufacturer will sell if it charges $9 for each bear.

\[ y = -4(9)^2 + 40(9) = 36 \]

9. Explain why a quadratic function models the data.

<table>
<thead>
<tr>
<th>x</th>
<th>2</th>
<th>4</th>
<th>6</th>
<th>8</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>f(x)</td>
<td>3</td>
<td>33</td>
<td>87</td>
<td>165</td>
<td>267</td>
</tr>
</tbody>
</table>

\[ \text{Diff} \]
\[ 30 \quad 54 \quad 78 \quad 102 \]

Increasing

\[ \text{Periodic Pattern} \]

10. Your class council determined that its profit from the upcoming homecoming dance is directly related to the ticket price for the dance. Looking at past dances, the council determined that the profit \( p \) can be modeled by the function \( p(t) = -12t^2 + 480t + 30 \), where \( t \) represents the price of each ticket. What should be the price of a ticket to the homecoming dance to maximize the council's profit?
11. A factory is producing a mirror in the shape of a parabola to be used in searchlights. A drawing of the mirror is shown. The light from the searchlight is located at the focus of the parabola, and will shine through at the given vertex of the mirror. Find an equation that represents the parabolic mirror.

\[ a = \frac{1}{4p} = \frac{1}{4(4)} = \frac{1}{16} \]

Because it's Concave Down.

\[ y = \frac{1}{16}(x-4)^2 + 3 \]

12. Describe the transformation of \( f(x) = x^2 \) represented by \( g \).

- Shifted down 2
- Vertical compression by \( \frac{1}{2} \)
- Translated right 4
- and up 2
- Reflected in X-Axis and down 2.

13. An object is launched directly overhead at 36 meters per second. The height (in meters) of the object is given by \( h(t) = -16t^2 + 36t + 5 \), where \( t \) is the time (in seconds) since the object was launched. For how many seconds is the object at or above a height of 25 meters?

14. A model rocket is launched from the top of a building. The height (in meters) of the rocket above the ground is given by \( h(t) = -6t^2 + 24t + 14 \), where \( t \) is the time (in seconds) since the rocket was launched. What is the rocket’s maximum height?