The average rate of change ("slope") is an exceptionally important concept in mathematics because it gives us a way to quantify how fast a function changes on average over a certain domain interval. Although we used its formula in the last exercise, we state it formally here:

\[ \text{Average Rate of Change ("slope")} \]

For a function over the domain interval \( a \leq x \leq b \), the function's average rate of change is calculated by:

\[ \frac{\Delta f}{\Delta x} = \frac{f(b) - f(a)}{b - a} \]

A linear function will have a \textit{constant} rate of change. *Know linear regressions \text{STAT} \rightarrow \text{CALC} \rightarrow \text{LinReg} (by)

Linear functions come in a variety of forms. The two shown below have been introduced in Common Core Algebra I and Common Core Geometry.

\[ \text{TWO COMMON FORMS OF A LINE} \]

- **Slope-Intercept**: \( y = mx + b \)
- **Point-Slope**: \( y - y_1 = m(x - x_1) \)

where \( m \) is the slope (or average rate of change) of the line and \( (x_1, y_1) \) represents one point on the line.

Transformations of Linear Functions:

- \( -f(x) = \) Reflection over \( x \)-axis
- \( f(-x) = \) Reflection over \( y \)-axis
- \( f(x) + k = \) Shifts up \( k \) units
- \( f(x) - k = \) Shifts down \( k \) units
- \( f(x + k) = \) Shifts left \( k \) units
- \( f(x - k) = \) Shifts right \( k \) units
- \( kf(x) = \) \textit{Vertically Stretched} by a factor of \( k \)
- \( \frac{1}{k} f(x) = \) \textit{Vertically Compressed} by a factor of \( k \)
- \( f(kx) = \) \textit{Horizontally Compressed} by a factor of \( k \)
- \( f\left(\frac{x}{k}\right) = \) \textit{Horizontally Stretched} by a factor of \( k \)
Solving Systems of Linear Equations

Systems of equations, or more than one equation, arise frequently in mathematics. To solve a system means to find all sets of values that simultaneously make all equations true.

Consider the three-by-three system of linear equations shown below. Each equation is numbered to help you keep track of the manipulations.

\[ \begin{align*}
2x + y + z &= 15 \\
6x - 3y - z &= 35 \\
-4x + 4y - z &= -14
\end{align*} \]

(a) The addition property of equality allows us to add two equations together to produce a third valid equation. Create a system by adding equations (1) and (2) and (1) and (3). Why is this an effective strategy in this case?

This is an effective strategy because in both cases the \( z \)-variable was eliminated from the equations.

\[ \begin{array}{c|c}
8x - 2y &= 50 \\
\hline
-8x + 20y &= 4
\end{array} \]

\[ \frac{8x - 2(3) = 50}{18y = 54} \]

\[ \frac{18y = 54}{y = 3} \]

\[ \begin{array}{c|c}
8x - 2(3) &= 50 \\
\hline
8x - 6 &= 50
\end{array} \]

\[ \frac{8x - 6 = 50}{8x = 56} \]

\[ \frac{8x = 56}{x = 7} \]

(b) Use this new two-by-two system to solve the three-by-three.

First multiply the second equation by 4:

\[ 4(-2x + 5y) = 4(1) \Rightarrow -8x + 20y = 4 \]

\[ \begin{align*}
2(7) + 3 + z &= 15 \\
z + 17 &= 15 \\
z &= -2
\end{align*} \]

Exercise #2: Consider the system of equations shown below. Answer the following questions based on the system.

\[ \begin{align*}
4x + y - 3z &= -6 \\
-2x + 4y + 2z &= 38 \\
5x - y - 7z &= -19
\end{align*} \]

The variable \( y \) will be easiest to eliminate first because it has a coefficient of 1 in the first equation.

\[ \begin{align*}
\begin{array}{c|c}
(1) + (3): \\
4x + y - 3z &= -6 \\
\hline
5x - y - 7z &= -19
\end{array} & \begin{array}{c|c}
-4x(1) + (2): \\
-16x - 4y + 12z &= 24 \\
\hline
-2x + 4y + 2z &= 38
\end{array} \\
\hline
9x - 10z &= -25 \\
\hline
-18x + 14z &= 62
\end{align*} \]

\[
\begin{align*}
18x - 20z &= -50 \\
-18x + 14z &= 62 \\
-6z &= 12 \\
\hline
z &= -2
\end{align*} \]

Multiply the first equation by 2:

\[
\begin{align*}
2(9x - 10z) &= 2(-25) \\
18x - 20z &= -50
\end{align*} \]

\[
\begin{align*}
18x - 20(-2) &= -50 \\
18x + 40 &= -50 \\
18x &= -90 \\
\hline
x &= -5
\end{align*} \]

\[
\begin{align*}
4(-5) + y - 3(-2) &= -6 \\
-20 + y + 6 &= -6 \\
y - 14 &= -6 \\
\hline
y &= 8
\end{align*} \]
QUADRATIC FUNCTIONS

Any function of the form \( f(x) = ax^2 + bx + c \) where the leading coefficient, \( a \), is not zero.

a) Standard Form: \( f(x) = ax^2 + bx + c \) useful for finding **Y-intercept** ("c" value)
b) Vertex Form: \( f(x) = a(x-h)^2 + k \) useful for finding **vertex** \((h, k)\)
c) Factored Form: \( f(x) = a(x-r_1)(x-r_2) \) useful for finding **roots** \( r_1, r_2 \)

Definitions:

1. Turning Point (Vertex): **The max or min of the parabola**
2. Axis of Symmetry: \( x = \frac{-b}{2a} \) (line down the middle of the parabola)
3. Focus: \( \frac{1}{4p} \) **these are the same**
4. Directrix: **line that the parabola bends away from 3 dist away from vertex ("p")**
5. Roots: \( x \) **its of the parabola**

Consider the simplest of all quadratic functions \( y = x^2 \).

(a) Create a table of values to plot this function over the domain interval \( -3 \leq x \leq 3 \).

<table>
<thead>
<tr>
<th>( x )</th>
<th>-3</th>
<th>-2</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y = x^2 )</td>
<td>9</td>
<td>4</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>4</td>
<td>9</td>
</tr>
</tbody>
</table>

(b) Sketch a graph of this function on the grid to the right.

(c) State the coordinates of the **turning point** of this parabola.

\( (0, 0) \)

(d) State the equation of this parabola's **axis of symmetry**.

\( x = 0 \)

(e) Over what interval is this function increasing? Decreasing?

\( 0 < x < \infty \) \quad \infty < x < 0 \)

How is \( x = y^2 \) different? Sketch the graph to the right.

State the Domain: \( x \geq 0 \)

State the Range: \( (-\infty, \infty) \)
**Example:** Use the formula \( x = \frac{-b}{2a} \) to find the turning points for each of the following quadratic functions.

(a) \( f(x) = 2x^2 - 12x + 7 \)

\[
\begin{align*}
x &= \frac{-(-12)}{2(2)} = \frac{12}{4} = 3 \\
y &= 2(3)^2 - 12(3) + 7 = -11 \\
\text{Turning point: } (3, -11)
\end{align*}
\]

Rewrite in Vertex form: \( y = 2(x-3)^2 - 11 \)

Describe the transformation from \( y = x^2 \)

- **Right 3 units**
- **Down 11 units**

(b) \( g(x) = -\frac{1}{4}x^2 + 5x - 20 \)

\[
\begin{align*}
x &= \frac{-5}{2(-\frac{1}{4})} = \frac{-5}{-\frac{1}{2}} = 10 \\
y &= -\frac{1}{4}(10)^2 + 5(10) - 20 = 5 \\
\text{Turning Point: } (10, 5)
\end{align*}
\]

Rewrite in Vertex form: \( y = -\frac{1}{4}(x-10)^2 + 5 \)

Describe the transformation from \( y = x^2 \)

- **Right 10 units**
- **Up 5 units**
- **Vert. Compressed by \( \frac{1}{4} \)**
- **(x-axis)**

**Quadratic Regressions:**

1. Hit STAT→EDIT. Enter data into L1, L2
2. Hit STAT→CALC. Select option #5 QuadReg
3. Scroll down to Calculate.
4. Copy down the equation. Round coefficients as specified.

**Example:**
A baseball is thrown up in the air. The table shows the heights \( y \) (in feet) of the baseball after \( x \) seconds.

<table>
<thead>
<tr>
<th>Time, ( x )</th>
<th>0</th>
<th>2</th>
<th>4</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>Baseball height, ( y )</td>
<td>6</td>
<td>22</td>
<td>22</td>
<td>6</td>
</tr>
</tbody>
</table>

\[ y = ax^2 + bx + c \]

\[
\begin{align*}
a &= -2 \\
b &= 12 \\
c &= 6
\end{align*}
\]

a) Write an equation for the path of the baseball.

\[ y = -2x^2 + 12x + 6 \]

b) Find the height of the baseball after 5 seconds.

16 ft

\[ y = 24 \text{ ft} \]

c) What is the maximum height of the baseball?

\[ z \text{ ft} \]

d) When does the baseball reach its maximum height? 3 sec's

**Focus & Directrix of a Parabola**

A parabola is the collection of all points **equidistant** from a fixed point (known as its **focus**) and a fixed line (known as its **directrix**).

Consider parabola that is the collection of all points equidistant from the point \( (0, 8) \) and the line \( y = 2 \).
(a) Give each of the following:

- **Directrix:** \( y = 2 \)
- **Focus:** \((0, 8)\)

(b) Draw a diagram of this parabola and label its turning point on the diagram below.

(c) Find the equation of this parabola.

(Use the formula \( a = \frac{1}{4p} \))

\[
y = a(x - 0)^2 + 5
\]

\[
y = \frac{1}{12}(x - 0)^2 + 5
\]

**Example 1:**

Determine the equation of the parabola whose focus is the point \((4, 1)\) and whose directrix is the horizontal line \(y = -3\). First, draw a diagram that shows the parabola, then carefully use the formula \( a = \frac{1}{4p} \) to derive its equation.

\[
y = a(x - 4)^2 - 1
\]

\[
p = 2 \Rightarrow \frac{1}{4a} = a = \frac{1}{8}
\]

\[
y = \frac{1}{8}(x - 4)^2 - 1
\]

**Example 2:**

Given the equation of a parabola to be \( y = \frac{1}{12}x^2 + 1 \), determine the following and sketch the graph labeling each of the parameters listed below:

- **a)** Vertex \((0, 1)\)
- **b)** "a" value \(\frac{1}{12}\)
- **c)** "p" value \(3\)
- **d)** Focus \((0, 4)\)
- **e)** Directrix \(y = -2\)

\[
a = \frac{1}{4p}
\]

\[
\frac{1}{12} = \frac{1}{4p} \Rightarrow p = 3
\]