Solutions for Section 4.4

Exercises

1. (a) Suppose $P$ is put in the account. The interest rate per month is 0.08/12. At the end of a year,
\[ \text{Balance} = \left(1 + \frac{0.08}{12}\right)^{12} = 1.08300, \]
which is 108.3% of the original amount. So the effective annual yield is 8.300%.
(b) With weekly compounding, the interest rate per week is 0.08/52. At the end of a year,
\[ \text{Balance} = \left(1 + \frac{0.08}{52}\right)^{52} = 1.08322, \]
which is 108.322% of the original amount. So the effective annual yield is 8.322%.
(c) Assuming it is not a leap year, the interest rate per day is 0.08/365. At the end of a year
\[ \text{Balance} = \left(1 + \frac{0.08}{365}\right)^{365} = 1.08328, \]
which is 108.328% of the original amount. So the effective annual yield is 8.328%.

2. The amount in the account at time $t$ is given by $1000b^t$. We set $1000b^{15} = 3500$ and solve for $b$:
\[ 1000b^{15} = 3500 \]
\[ b^{15} = 3.5 \]
\[ b = (3.5)^{1/15} = 1.0871. \]
The effective annual yield over the 15-year period was 8.71% per year.

3. If $P$ is the initial amount, the amount after 20 years is $P(1.05)^{20} = P(2.653)$. Since 2.653 = 1 + 1.653, the investment has increased by 165.3% over the 20-year period.

4. If $P$ is the initial amount, the amount after 8 years is 0.5$P$. To find the effective annual yield, we set $Pb^8$ equal to 0.5$P$ and solve for $b$:
\[ b^8 = 0.5 \]
\[ b = (0.5)^{1/8} = 0.917. \]
Since 0.917 = 1 – 0.083, the investment has decreased by an effective annual rate of −8.3% per year.

5. (a) The nominal interest rate is 8%, so the interest rate per month is 0.08/12. Therefore, at the end of 3 years, or 36 months,
\[ \text{Balance} = 1000 \left(1 + \frac{0.08}{12}\right)^{36} = 1270.24. \]
(b) There are 52 weeks in a year, so the interest rate per week is 0.08/52. At the end of 52 × 3 = 156 weeks,
\[ \text{Balance} = 1000 \left(1 + \frac{0.08}{52}\right)^{156} = 1271.01. \]
(c) Assuming no leap years, the interest rate per day is 0.08/365. At the end of 3 × 365 days
\[ \text{Balance} = 1000 \left(1 + \frac{0.08}{365}\right)^{3 \times 365} = 1271.22. \]

6. (a) $B = B_0(1.013)^{1} = B_0(1.013)$, so the effective annual rate is 1.3%.
(b) $B = B_0 \left(1 + \frac{0.013}{12}\right)^{12} = B_0(1.0131)$, so the effective annual rate is approximately 1.31%.
7. (a) If the interest is compounded annually, there will be $500 \cdot 1.01 = $505 after one year.
    (b) If the interest is compounded weekly, after one year, there will be $500 \cdot (1 + 0.01/52)^{52} = $505.02.
    (c) If the interest is compounded every minute, after one year, there will be $500 \cdot (1 + 0.01/52,600)^{52,600} = $505.03.

8. (a) If the interest is compounded annually, there will be $500 \cdot 1.03 = $515 after one year.
    (b) If the interest is compounded weekly, there will be $500 \cdot (1 + 0.03/52)^{52} = $515.22 after one year.
    (c) If the interest is compounded every minute, there will be $500 \cdot (1 + 0.03/52,600)^{52,600} = $515.23 after one year.

9. (a) If the interest is compounded annually, there will be $500 \cdot 1.05 = $525 after one year.
    (b) If the interest is compounded weekly, there will be $500 \cdot (1 + 0.05/52)^{52} = $525.62 after one year.
    (c) If the interest is compounded every minute, there will be $500 \cdot (1 + 0.05/52,600)^{52,600} = $525.64 after one year.

10. (a) If the interest is compounded annually, there will be $500 \cdot 1.08 = $540 after one year.
     (b) If the interest is compounded weekly, there will be $500 \cdot (1 + 0.08/52)^{52} = $541.61 after one year.
     (c) If the interest is compounded every minute, there will be $500 \cdot (1 + 0.08/52,600)^{52,600} = $541.64 after one year.

11. (a) The nominal rate is the stated annual interest without compounding, thus 1%.
    The effective annual rate for an account paying 1% compounded annually is 1%.
    (b) The nominal rate is the stated annual interest without compounding, thus 1%.
    With quarterly compounding, there are four interest payments per year, each of which is $1/4 = 0.25\%$. Over the course of the year, this occurs four times, giving an effective annual rate of $1.0025^4 = 1.01004$, which is 1.004%.
    (c) The nominal rate is the stated annual interest without compounding, thus 1%.
    With daily compounding, there are 365 interest payments per year, each of which is $(1/365)\%$. Over the course of the year, this occurs 365 times, giving an effective annual rate of $(1 + 0.01/365)^{365} = 1.01005$, which is 1.005%.

12. (a) The nominal rate is the stated annual interest without compounding, thus 100%.
    The effective annual rate for an account paying 1% compounded annually is 100%.
    (b) The nominal rate is the stated annual interest without compounding, thus 100%.
    With quarterly compounding, there are four interest payments per year, each of which is $100/4 = 25\%$. Over the course of the year, this occurs four times, giving an effective annual rate of $1.25^4 = 2.44141$, which is 144.141%.
    (c) The nominal rate is the stated annual interest without compounding, thus 100%.
    With daily compounding, there are 365 interest payments per year, each of which is $(100/365)\%$. Over the course of the year, this occurs 365 times, giving an effective annual rate of $(1 + 1/365)^{365} = 2.71457$, which is 171.457%.

13. (a) The nominal rate is the stated annual interest without compounding, thus 3%.
    The effective annual rate for an account paying 1% compounded annually is 3%.
    (b) The nominal rate is the stated annual interest without compounding, thus 3%.
    With quarterly compounding, there are four interest payments per year, each of which is $3/4 = 0.75\%$. Over the course of the year, this occurs four times, giving an effective annual rate of $1.0075^4 = 1.03034$, which is 3.034%.
    (c) The nominal rate is the stated annual interest without compounding, thus 3%.
    With daily compounding, there are 365 interest payments per year, each of which is $(3/365)\%$. Over the course of the year, this occurs 365 times, giving an effective annual rate of $(1 + 0.03/365)^{365} = 1.03045$, which is 3.045%.

14. (a) The nominal rate is the stated annual interest without compounding, thus 6%.
    The effective annual rate for an account paying 1% compounded annually is 6%.
    (b) The nominal rate is the stated annual interest without compounding, thus 6%.
    With quarterly compounding, there are four interest payments per year, each of which is $6/4 = 1.5\%$. Over the course of the year, this occurs four times, giving an effective annual rate of $1.015^4 = 1.06156$, which is 6.136%.
    (c) The nominal rate is the stated annual interest without compounding, thus 6%.
    With daily compounding, there are 365 interest payments per year, each of which is $(6/365)\%$. Over the course of the year, this occurs 365 times, giving an effective annual rate of $(1 + 0.06/365)^{365} = 1.06183$, which is 6.183%.

Problems

15. If the investment is growing by 3% per year, we know that, at the end of one year, the investment will be worth 103% of what it had been the previous year. At the end of two years, it will be 103% of 103% = (1.03)^2 as large. At the end of 10 years, it will have grown by a factor of $(1.03)^{10}$, or 1.34392. The investment will be 134.392% of what it had been, so
we know that it will have increased by 34.392%. Since \((1.03)^{10} \approx 1.34392\), it increases by 34.392%.

16. If the annual growth factor is \(b\), then we know that, at the end of 5 years, the investment will have grown by a factor of \(b^5\). But we are told that it has grown by 30%, so it is 130% of its original size. So

\[
\begin{align*}
b^5 &= 1.30 \\
b &= 1.30^{\frac{1}{5}} \approx 1.06387.
\end{align*}
\]

Since the investment is 106.387% as large as it had been the previous year, we know that it is growing by about 5.387% each year.

17. Let \(b\) represent the growth factor, since the investment decreases, \(b < 1\). If we start with an investment of \(P_0\), then after 12 years, there will be \(P_0b^{12}\) left. But we know that since the investment has decreased by 60% there will be 40% remaining after 12 years. Therefore,

\[
\begin{align*}
P_0b^{12} &= P_00.40 \\
b^{12} &= 0.40 \\
b &= (0.40)^{\frac{1}{12}} \approx 0.92648.
\end{align*}
\]

This tells us that the value of the investment will be 92.618% of its value the previous year, or that the value of the investment decreases by approximately 7.382% each year, assuming a constant percent decay rate.

18. (a) Let \(x\) be the amount of money you will need. Then, at 5% annual interest, compounded annually, after 6 years you will have the following dollar amount:

\[x(1 + 0.05)^6 = x(1.05)^6.\]

If this needs to equal $25,000, then we have

\[
x(1.05)^6 = 25,000 \\
x = \frac{25,000}{(1.05)^6} \approx 18,655.38.
\]

(b) At 5% annual interest, compounded monthly, after 6 years, or \(6 \cdot 12 = 72\) months, you will have the following dollar amount:

\[x \left(1 + \frac{0.05}{12}\right)^{72}.\]

If this needs to equal $25,000, then we have

\[
x \left(1 + \frac{0.05}{12}\right)^{72} = 25,000 \\
x = \frac{25,000}{\left(1 + \frac{0.05}{12}\right)^{72}} \approx 18,532.00.
\]

(c) At 5% annual interest, compounded daily, after 6 years, or \(6 \cdot 365 = 2190\) days, you will have the following dollar amount:

\[x \left(1 + \frac{0.05}{365}\right)^{2190} = x(1.000136986)^{2190}.\]

If this needs to equal $25,000, then we have

\[x(1.000136986)^{2190} = 25,000 \\
x = \frac{25,000}{(1.000136986)^{2190}} \approx 18,520.84.
\]

(d) The effective yield on an account increases with the number of times of compounding. So, as the number of times increases, the amount of money you need to begin with in order to end up with $25,000 in 6 years decreases.
19. Let \( r \) represent the nominal annual rate. Since the interest is compounded quarterly, the investment earns \( \frac{r}{4} \) each quarter. So, at the end of the first quarter, the investment is \( 850 \left(1 + \frac{r}{4}\right) \), and at the end of the second quarter is \( 850 \left(1 + \frac{r}{4}\right)^2 \). By the end of 40 quarters (which is 10 years), it is \( 850 \left(1 + \frac{r}{4}\right)^{40} \). But we are told that the value after 10 years is $1,000, so

\[
1000 = 850 \left(1 + \frac{r}{4}\right)^{40}
\]

\[
\frac{1000}{850} = \left(1 + \frac{r}{4}\right)^{40}
\]

\[
\frac{20}{17} = \left(1 + \frac{r}{4}\right)^{40}
\]

\[
\left(\frac{20}{17}\right)^{\frac{1}{40}} = 1 + \frac{r}{4}
\]

\[
1.00407 \approx 1 + \frac{r}{4}
\]

\[
0.00407 \approx \frac{r}{4}
\]

\[
r \approx 0.01628.
\]

We see that the nominal interest rate is 1.628%.

20. (a) The effective annual rate is the rate at which the account is actually increasing in one year. According to the formula, \( M = M_0(1.07763)^t \), at the end of one year you have \( M = 1.07763M_0 \), or 1.07763 times what you had the previous year. The account is 107.763% larger than it had been previously; that is, it increased by 7.763%. Thus the effective rate being paid on this account each year is about 7.763%.

(b) Since the money is being compounded each month, one way to find the nominal annual rate is to determine the rate being paid each month. In \( t \) years there are \( 12t \) months, and so, if \( b \) is the monthly growth factor, our formula becomes

\[
M = M_0b^{12t} = M_0(b^{12})^t.
\]

Thus, equating the two expressions for \( M \), we see that

\[
M_0(b^{12})^t = M_0(1.07763)^t.
\]

Dividing both sides by \( M_0 \) yields

\[
(b^{12})^t = (1.07763)^t.
\]

Taking the \( t^{th} \) root of both sides, we have

\[
b^{12} = 1.07763
\]

which means that

\[
b = (1.07763)^{1/12} \approx 1.00625.
\]

Thus, this account earns 0.625% interest every month, which amounts to a nominal interest rate of about 12(0.625%) = 7.5%.

21. (i) Equation (b). Since the growth factor is 1.12, or 112%, the annual interest rate is 12%.

(ii) Equation (a). An account earning at least 1% monthly will have a monthly growth factor of at least 1.01, which means that the annual (12-month) growth factor will be at least

\[
(1.01)^{12} \approx 1.1268.
\]

Thus, an account earning at least 1% monthly will earn at least 12.68% yearly. The only account that earns this much interest is account (a).

(iii) Equation (c). An account earning 12% annually compounded semi-annually will earn 6% twice yearly. In \( t \) years, there are 2\( t \) half-years.

(iv) Equations (b), (c) and (d). An account that earns 3% each quarter ends up with a yearly growth factor of \((1.03)^4 \approx 1.1255\). This corresponds to an annual percentage rate of 12.55%. Accounts (b), (c) and (d) earn less than this. Check this by determining the growth factor in each case.
(v) Equations (a) and (e). An account that earns 6% every 6 months will have a growth factor, after 1 year, of \((1 + 0.06)^2 = 1.1236\), which is equivalent to a 12.36% annual interest rate, compounded annually. Account (a), earning 20% each year, clearly earns more than 6% twice each year, or 12.36% annually. Account (e), which earns 3% each quarter, earns \((1.03)^3 = 1.0909\), or 6.99% every 6 months, which is greater than 6%.

22. (a) The investment is initially worth $1000, and it grows in value by 7.7% each year.
(b) The investment is initially worth $9500, but it loses value by 5.5% each year.
(c) The investment is initially worth $1000, and it triples in value once every five years.
(d) The investment is initially worth $500, and it earns 4% annual interest, compounded monthly.

**Solutions for Section 4.5**

**Skill Refresher**

S1. We have \(e^{0.07} = 1.073\).
S2. We have \(10e^{-0.14} = 8.694\).
S3. We have \(\frac{2}{\sqrt{e}} = 1.433\).
S4. We have \(e^{3e} = 3480.202\).
S5. We have \(f(0) = 2.3e^{0.3(0)} = 2.3\) and \(f(4) = 2.3e^{0.3(4)} = 7.636\).
S6. We have \(g(0) = 4.2e^{-0.12(0)} = 4.2\) and \(g(4) = 4.2e^{-0.12(4)} = 2.599\).
S7. We have \(h(0) = 153 + 8.6e^{0.43(0)} = 153 + 8.6 = 161.6\) and \(h(4) = 153 + 8.6e^{0.43(4)} = 153 + 48.027 = 201.027\).
S8. We have \(k(0) = 289 - 4.7e^{-0.0018(0)} = 289 - 4.7 = 284.3\) and \(k(4) = 289 - 4.7e^{-0.0018(4)} = 289 - 4.666 = 284.334\).
S9. Writing the function as

\[
    f(t) = (3e^{0.04}(t))^{3} = 3^{3}e^{0.04(t)} = 27e^{0.12t},
\]

we have \(a = 27\) and \(k = 0.12\).
S10. Writing the function as

\[
    g(x) = 5e^{x^{2}} \cdot e^{4x} \cdot 3e^{x} = (5 \cdot 3)e^{x^{2}+4x+1} = 15e^{12x},
\]

we have \(a = 15\) and \(k = 12\).
S11. To convert the form \(Q = ae^{kt}\), we use the property that \(e^{m+n} = e^{m} \cdot e^{n}\). Thus, we have \(Q = e^{7} \cdot e^{-3t} = 1.096633e^{-3t}\).
S12. To convert the form \(Q = ae^{kt}\), we first use the property that \((e^m)^n = e^{mn}\).

\[
    Q = \sqrt{e^{3+6t}} = (e^{3+6t})^{1/2} = e^{3/2} \cdot e^{3t} = e^{3/2+3t}.
\]

Next we use the property that \(e^{m+n} = e^{m} \cdot e^{n}\). Thus, we have \(Q = e^{3/2+3t} = e^{3/2} \cdot e^{3t} = 4.482e^{3t}\).
S13. Writing the function as

\[
    m(x) = \frac{7e^{0.2x}}{\sqrt{3e^{x}}} = \frac{7}{\sqrt{3}} e^{0.5x} = \frac{7}{\sqrt{3}} e^{-0.5x} = \frac{7}{\sqrt{3}} e^{-0.3x},
\]

we have \(a = \frac{7}{\sqrt{3}}\) and \(k = -0.3\).
S14. Writing the function as

\[
    P(t) = \left(2 \sqrt{e^{5t}}\right)^{4} = 2^{4} \left(e^{5t}\right)^{4} = 16e^{20t} \cdot e^{20t} = 16e^{20t},
\]

we have \(a = 16\) and \(k = 20/3\).