Section 10.1 – Function Composition

The function \( h(t) = f(g(t)) \) is called the composition of \( f \) with \( g \). The function \( h \) is defined by using the output of the function \( g \) as the input of \( f \).

Example 1. Complete the table below.

<table>
<thead>
<tr>
<th></th>
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<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>( t )</td>
<td>( f(t) )</td>
<td>2</td>
<td>3</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>( g(t) )</td>
<td>3</td>
<td>1</td>
<td>2</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>( f(g(t)) )</td>
<td>1</td>
<td>3</td>
<td>1</td>
<td>2</td>
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<tr>
<td>( g(f(t)) )</td>
<td>2</td>
<td>0</td>
<td>1</td>
<td>1</td>
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</tbody>
</table>

Example 2. Let \( f(x) = x^2 - 1 \), \( g(x) = \frac{2x^2}{x - 1} \), and \( p(x) = \sqrt{x} \). Find and simplify each of the following.

(a) \( g(f(x)) \)

\[
g(f(x)) = g(x^2 - 1) = \frac{(x^2 - 1)^2}{(x^2 - 1) - 1} = \frac{2(x^4 - 2x^2 + 1)}{x^2 - 2} = \frac{2x^4 - 4x^2 + 2}{x^2 - 2}.
\]

(b) \( p(g(x^2)) \)

\[
p(g(x^2)) = p(\frac{2(x^2)^2}{x^2 - 1}) = \sqrt{\frac{2x^4}{x^2 - 1}} = \frac{x\sqrt{2}}{\sqrt{x^2 - 1}}.
\]

(c) First, note that \( f(x) = x^2 - 1 \), so \( f(x + h) = (x + h)^2 - 1 = x^2 + 2xh + h^2 - 1 \). Therefore, we have

\[
\frac{f(x + h) - f(x)}{h} = \frac{(x^2 + 2xh + h^2 - 1) - (x^2 - 1)}{h} = \frac{x^2 + 2xh + h^2 - 1 - x^2 + 1}{h} = \frac{2xh + h^2}{h} = \frac{h(2x + h)}{h} = 2x + h,
\]

so our final answer is \( 2x + h \).
Example 3. For the function \( f(x) = (x^3 + 1)^2 \), find functions \( u(x) \) and \( v(x) \) such that \( f(x) = u(v(x)) \).

First, we let \( v(x) = x^3 + 1 \), the "inside" function. Then \( f(x) = (x^3 + 1)^2 = (v(x))^2 \), so we see that the function \( v(x) \) is being squared to obtain \( f(x) \). Therefore, \( u(x) = x^2 \). To see that our answer is right, note that
\[
 u(v(x)) = (v(x))^2 = (x^3 + 1)^2 = f(x).
\]
Thus, our answers are \( u(x) = x^2 \) and \( v(x) = x^3 + 1 \).

Examples and Exercises

1. Given to the right are the graphs of two functions, \( f \) and \( g \). Use the graphs to estimate each of the following.

   (a) \( g(f(0)) = -2.3 \) 
   \( g(f(0)) = g(2.8) = -2.3 \) 
   (b) \( f(g(0)) = 2.4 \) 
   \( f(g(0)) = f(-1) = 2.4 \)

   (c) \( f(g(3)) = 0.9 \) 
   \( f(g(3)) = f(-2.5) = 0.9 \) 
   (d) \( g(g(4)) = 0.5 \) 
   \( g(g(4)) = g(-3) = 0.5 \)

   (e) \( f(f(1)) = 1.3 \) 
   \( f(f(1)) = f(2.8) = 1.3 \)

2. For each of the following functions \( f(x) \), find functions \( u(x) \) and \( v(x) \) such that \( f(x) = u(v(x)) \).

   (a) \( \sqrt{1 + x} \)

   Let \( v(x) = 1 + x \) be the "inside" function.
   Then \( \sqrt{1 + x} = \sqrt{v(x)} \), so we need to take the square root of \( v(x) \) to get \( f(x) = \sqrt{1 + x} \). Therefore, \( u(x) = \sqrt{x} \). To check our answer, note that
\[
 u(v(x)) = u(1 + x) = \sqrt{1 + x} = f(x),
\]
so our final answer is \( u(x) = \sqrt{x} \) and \( v(x) = 1 + x \).

   (b) \( \sin(x^3 + 1) \cos(x^3 + 1) \)

   Let \( v(x) = x^3 + 1 \) be the "inside" function. Then
\[
 \sin(x^3 + 1) \cos(x^3 + 1) = \sin(v(x)) \cos(v(x)),
\]
so we need to substitute \( v(x) \) into the function \( u(x) = \sin x \cos x \) to get
\[
 f(x) = \sin(x^3 + 1) \cos(x^3 + 1).
\]
To check our answer, note that
\[
 u(v(x)) = u(x^3 + 1) = \sin(x^3 + 1) \cos(x^3 + 1) = f(x),
\]
so our final answer is \( u(x) = \sin x \cos x \) and \( v(x) = x^3 + 1 \).
(c) \(3^{2x+1}\)

Let \(v(x) = 2x + 1\), so that \(3^{2x+1} = 3^v(x)\).

Therefore, if we choose \(u(x) = 3^x\), we have

\[ u(v(x)) = u(2x + 1) = 3^{2x+1} \]

as desired. Our final answers are therefore \(u(x) = 3^x\) and \(v(x) = 2x + 1\).

3. Let \(f(x) = \frac{1}{1 + 2x}\).

(a) Solve \(f(x + 1) = 4\) for \(x\).

We have

\[
f(x + 1) = 4 \implies \frac{1}{1 + 2(x + 1)} = 4 \implies \frac{1}{2x + 3} = 4 \implies 1 = 4(2x + 3)
\]

\[= 1 = 8x + 12 \]

\[\implies 8x = -11,
\]

so our final answer is \(x = -11/8\).

(b) Solve \(f(x) + 1 = 4\) for \(x\).

We have

\[
f(x) + 1 = 4 \implies f(x) = 3 \implies \frac{1}{1 + 2x} = 3 \implies 1 = 3(1 + 2x)
\]

\[\implies 1 = 3 + 6x \]

\[\implies 6x = -2,
\]

so our answer is \(x = -1/3\).

(c) Calculate \(f(f(x))\) and simplify your answer.

We have

\[
f(f(x)) = f\left(\frac{1}{1 + 2x}\right) = \frac{1}{1 + 2\left(\frac{1}{1+2x}\right)} = \frac{1}{1 + \frac{2}{1+2x}}
\]

\[= \frac{1}{\frac{1+2x}{1+2x} + \frac{2}{1+2x}}
\]

\[= \frac{1}{\frac{1+2x+2}{1+2x}}
\]

\[= \frac{1 \cdot 1 + 2x}{1 \cdot 3 + 2x}
\]

so our final answer is \(\frac{1+2x}{3+2x}\).
4. For each of the following functions, calculate
\[
\frac{f(x + h) - f(x)}{h}
\]
and simplify your answers.

(a) \( f(x) = x^2 + 2x + 1 \)  \hspace{1cm} (b) \( f(x) = \frac{1}{x} \)  \hspace{1cm} (c) \( f(x) = 3x + 1 \)

(a)
\[
\frac{f(x + h) - f(x)}{h} = \frac{(x + h)^2 + 2(x + h) + 1 - (x^2 + 2x + 1)}{h} = \frac{x^2 + 2xh + h^2 + 2x + 2h + 1 - x^2 - 2x - 1}{h} = \frac{2xh + h^2 + 2h}{h} = \frac{h(2x + h + 2)}{h},
\]
so after canceling the factor of \( h \), our final answer is \( 2x + h + 2 \).

(b)
\[
\frac{f(x + h) - f(x)}{h} = \frac{\frac{1}{x + h} - \frac{1}{x}}{h} = \frac{\frac{x - (x + h)}{x(x + h)}}{h} = \frac{-h}{x(x + h)} \cdot \frac{1}{h},
\]
so after canceling the factor of \( h \), our final answer is \( -\frac{1}{x(x + h)} \).

(c)
\[
\frac{f(x + h) - f(x)}{h} = \frac{3(x + h) + 1 - (3x + 1)}{h} = \frac{3x + 3h + 1 - 3x - 1}{h} = \frac{3h}{h} = 3,
\]
so our final answer is simply 3.
Section 10.2 – Invertibility and Properties of Inverse Functions

**Definition.** Suppose \( Q = f(t) \) is a function with the property that each value of \( Q \) determines exactly one value of \( t \). Then \( f \) has an inverse function, \( f^{-1} \), and

\[
f^{-1}(Q) = t \quad \text{if and only if} \quad Q = f(t).
\]

If a function has an inverse, it is said to be invertible.

**Example 1.** Given below are values for a function \( Q = f(t) \). Fill in the corresponding table for \( t = f^{-1}(Q) \).

<table>
<thead>
<tr>
<th>( t )</th>
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<tbody>
<tr>
<td>( f(t) )</td>
<td>2</td>
<td>5</td>
<td>7</td>
<td>8</td>
<td>11</td>
</tr>
</tbody>
</table>

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<th>8</th>
<th>11</th>
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</tbody>
</table>

**Observation:**

\[
f^{-1}(f(2)) = f^{-1}(7) = 2 \quad f(f^{-1}(5)) = f(1) = 5
\]

**Comment:** In general, \( f(f^{-1}(x)) = x \) and \( f^{-1}(f(x)) = x \).

**Question.** Does the function \( f(x) = x^2 \) have an inverse function?

No, because (for example) \( f(1) = 1 \) and \( f(-1) = 1 \). Therefore, if there were a function \( f^{-1} \), we would have \( f^{-1}(1) \) equal to both 1 and -1, which is impossible.

**Comment:** Note that this example, when interpreted graphically, illustrates that \( f(x) = x^2 \) fails the Horizontal Line Test.

**Horizontal Line Test.** A function \( f \) has an inverse function if and only if the graph of \( f \) intersects any horizontal line at most once. In other words, if any horizontal line touches the graph of \( f \) in more than one place, then \( f \) is not invertible.
Example 2. Suppose \( B = f(t) = 5(1.04)^t \), where \( B \) is the balance in a bank account, in thousands of dollars, after \( t \) years.

(a) Find a formula for the inverse function of \( f \).

We are given \( B \) as a function of \( t \), so finding the inverse amounts to finding \( t \) as a function of \( B \). We have

\[
\begin{align*}
B &= 5(1.04)^t & \text{← } B \text{ as a function of } t \\
\frac{B}{5} &= 1.04^t \\
\ln(B/5) &= t \ln 1.04 \\
t &= \frac{\ln(B/5)}{\ln 1.04} & \text{← } t \text{ as a function of } B
\end{align*}
\]

Therefore, the formula for the inverse function is \( f^{-1}(B) = \frac{\ln(B/5)}{\ln 1.04} \).

(b) Compute each of the following and interpret them practically: (i) \( f(20) \) (ii) \( f^{-1}(20) \)

(i) \( f(20) = 5(1.04)^{20} \approx 10.96 \) thousand dollars is the amount of money in the account after 20 years.

(ii) Using the answer to part (a) above, we have

\[
f^{-1}(20) = \frac{\ln(20/5)}{\ln 1.04} \approx 35.35 \text{ years}.
\]

Therefore, \( f^{-1}(20) = 35.35 \text{ years} \) is the amount of time it takes until the account has 20 thousand dollars in it.
Examples and Exercises

1. Find a formula for the inverse function of each of the following functions.

(a) \( f(x) = \frac{x - 1}{x + 1} \)

We have

\[
y = \frac{x - 1}{x + 1} \implies y(x + 1) = x - 1 \implies yx + y = x - 1
\]

\[
\implies yx - x = -1 - y
\]

\[
\implies x(y - 1) = -1 - y,
\]

so \( x = \frac{-1 - y}{y - 1} \), and our answer is therefore \( f^{-1}(y) = \frac{-1 - y}{y - 1} \).

(b) \( g(x) = \ln(3 - x) \)

We have

\[
y = \ln(3 - x) \implies e^y = e^{\ln(3-x)} \implies e^y = 3 - x
\]

\[
\implies x = 3 - e^y,
\]

so our answer is \( g^{-1}(y) = 3 - e^y \).

2. Given to the right is the graph of the functions \( f(x) \) and \( g(x) \). Use the function to estimate each of the following.

(a) \( f(2) = -1 \)  
(b) \( f^{-1}(2) = -4 \)

(c) \( f^{-1}(g(-1)) \approx 2.6 \)
\( f^{-1}(g(-1)) \approx f^{-1}(-2) \approx 2.6 \)

(d) \( g^{-1}(f(3)) = -2.7 \)
\( g^{-1}(f(3)) \approx g^{-1}(-3.2) \approx -2.7 \)

(e) Rank the following quantities in order from smallest to largest: \( f(1), f(-2), f^{-1}(1), f^{-1}(-2), 0 \)

\[ f^{-1}(1) < f(1) < 0 < f(-2) < f^{-1}(-2) \]
3. Let \( f(x) = 10e^{(x-1)/2} \) and \( g(x) = 2\ln x - 2\ln 10 + 1 \). Show that \( g(x) \) is the inverse function of \( f(x) \).

We have

\[
f(g(x)) = f(2\ln x - 2\ln 10 + 1) = 10e^{(2\ln x - 2\ln 10 + 1 - 1)/2} \\
= 10e^{2\ln x - \ln 10}/2 \\
= 10e^{\ln x - \ln 10} \\
= 10\cdot(x/10) = x,
\]

so \( f(g(x)) = x \) for all \( x \). Similarly, we have

\[
g(f(x)) = 2\ln(10e^{(x-1)/2}) - 2\ln 10 + 1 \\
= 2(\ln 10 + \ln e^{(x-1)/2}) - 2\ln 10 + 1 \\
= 2\ln 10 + 2\left(\frac{x-1}{2}\right) - 2\ln 10 + 1 \\
= 2\ln 10 + (x - 1) - 2\ln 10 + 1 \\
= x,
\]

so \( g(f(x)) = x \) for all \( x \). Therefore, since \( f(g(x)) = g(f(x)) = x \) for all \( x \), we see that \( g(x) = f^{-1}(x) \).

4. Let \( f(t) \) represent the amount of a radioactive substance, in grams, that remains after \( t \) hours have passed. Explain the difference between the quantities \( f(8) \) and \( f^{-1}(8) \) in the context of this problem.

\( f(8) \) is the amount, in grams, of the radioactive substance that remains after 8 hours.

On the other hand, \( f^{-1}(8) \) is the time that it takes, in hours, until only 8 grams of the radioactive substance remains.